Practice problems for (the limits portion of) Exam I

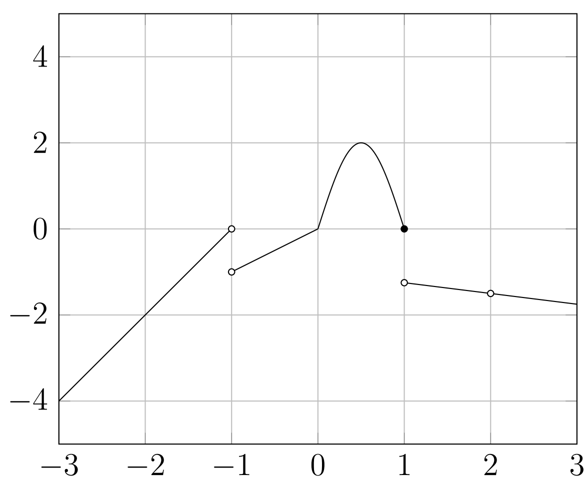
1 Limits, graphically

Exercise 1.1. Draw the graph of a function $f(x)$ such that $f(x)$ is not defined at 2.

Exercise 1.2. Draw the graph of a function $f(x)$ such that $f(x)$ has a righthand limit and a lefthand limit at 3, but for which these one-sided limits do not agree.

Exercise 1.3. Draw the graph of a function that does not have a limit at -2, but is defined at -2.

Exercise 1.4. Below is the graph of a function $f(x)$.



State whether or not the following limits exist. When they exist, state the value of the limit (you may estimate based on the graph).

(a) $\lim_{x\to -1} f(x)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1^-} f(x)$

(d) $\lim_{x\to 0} f(x)$

(e) $\lim_{x \to 0^-} f(x)$

(f) $\lim_{x \to 0^+} f(x)$

(g) $\lim_{x \to 1^-} f(x)$

(h) $\lim_{x \to 1^+} f(x)$

(i) $\lim_{x \to 1} f(x)$

(j) $\lim_{x \to 2^-} f(x)$

(k) $\lim_{x \to 2^+} f(x)$

(l) $\lim_{x \to 2} f(x)$

Finally, answer the following questions:

(m) Is $f(x)$ defined at 0?

(n) Is $f(x)$ continuous at 0?

(o) Is $f(x)$ defined at 1?

(p) Is $f(x)$ continuous at 1?

(q) Is $f(x)$ defined at -1?

(r) Is $f(x)$ continuous at -1?

(s) Is $f(x)$ defined at 2?

(t) Is $f(x)$ continuous at 2?

2 Limits and proofs

Exercise 2.1. State the ϵ-δ definition of the statement

$$\lim_{x\to a} f(x) = L.$$

Exercise 2.2. Below is a proof of the root law:

Let $g(x) = x^{1/n}$. *Then*

$$\begin{aligned}\lim_{x\to a}(f(x))^{1/n} &= \lim_{x\to a} g(f(x)) &(14)\\ &= g(\lim_{x\to a} f(x)) &(15)\\ &= (\lim_{x\to a} f(x))^{1/n}. &(16)\end{aligned}$$

In which lines were the following facts used?

(a) The composition law.

(b) The fact that $g(x) = x^{1/n}$ is continuous.

(c) The definition of $g(x)$.

(For example, (c) was used in the very first equality, (14). In fact, (c) is used in a later line, too!)

Exercise 2.3. Let $f(x) = x^2 + 3$. Let $L = 12$. Show that for any $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\text{if } 0 < |x - 3| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Exercise 2.4. Let $f(x) = \frac{x^2-4}{x+2}$. Let $L = 3$. Show that for any $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\text{if } 0 < |x - 5| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Exercise 2.5. Let $f(x) = \frac{x^2-4}{x-2}$. Using the ϵ-δ definition of limit, prove that

$$\lim_{x\to -2} f(x) = 0.$$

3 Continuity

Exercise 3.1. Let $f(x)$ be a function. State the *three conditions* you need to check to verify that $f(x)$ is continuous at a.

You may assume that all the "standard" functions we know are continuous. For example,

1. Constant functions
2. Polynomials
3. Trigonometric functions
4. Exponential functions
5. Logarithmic functions

are all continuous.

Exercise 3.2. State the intermediate value theorem.

Exercise 3.3. Let $f(x) = x^5 - 13$. Using the intermediate value theorem, prove that $f(x)$ has a root somewhere between $x = 1$ and $x = 2$.

4 Computing limits algebraically

Below, you do *not* need to use the ϵ-δ definitions of limit. Instead, you may freely use the limit laws. If asked, you should be able to say which limit laws you are using, and where. If asked, you should be able to say if, and where, you are invoking the continuity of certain standard functions.

Exercise 4.1. Compute the following limits; if the limit does not exist, state that the limit does not exist.

(a) $\lim_{x\to 3} x^3$

(b) $\lim_{x\to 0} \frac{x^3}{x}$

(c) $\lim_{x\to 2} \frac{x^3-x}{x^2-1}$

(d) $\lim_{x\to 0} \frac{\sin(x)+1}{\cos(x)}$

(e) $\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$

(f) $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

(g) You are told that $\lim_{x\to 2} g(x) = \pi$. Compute $\lim_{x\to 2}(\sin \circ g)(x)$.

(h) You are told that $\lim_{x\to -1} f(x) = 2$. Compute $\lim_{x\to -1} \left(\sqrt{f(x)}\right)$.