Derivative practice problems for Exam I

This is the final batch of practice problems for Exam I.

Make sure to look at all the problems on previous quizzes, and that you can do them. Each preparation packet has a few problems for you to practice, too. Look over also the problems from lab.

1 Computing derivatives

1.1

Let f be a function. State the definition of the derivative of f at x.

Warning: The definition is NOT the "slope of the tangent line." As I've said in class, that is a geometric *intuition,* and it is not the *definition.*

1.2

For each function $f(x)$ below, tell me what $f'(x)$ is.

(a) $\sin(x)$

(b) $\cos(x)$

(c) e^x

(d) $\ln x$

(e) x

1.3

Compute the derivatives of the following functions:

(a) $x^3 + 9x^2 + 18$

(b) $9x\sin(x)$

(c) $\sin(e^x)$

(d) $e^{\sin(x)}$

(e) $\sin(x^3 + 3x + 9)$

(f) $\frac{\sin(x)}{x^2+1}$

(g) $e^x \cdot \sin(x)$

(h) $\sqrt{\sqrt{e^x}}$

(i) 8^x

2 Critical points and applications

2.1

(a) State the definition of a critical point. That is, x is a critical point of f if....?

(b) State the definition of an inflection point.

2.2

Find the absolute minima and maxima of the following functions in the indicated intervals:

(a) $f(x) = x^2 + 3x + 1$ in the interval $[-1, 3]$

(b) $f(x) = xe^x$ in the interval $[-8, 8]$.

2.3

State where all the inflection points are of the following functions:

(a) $\tan x$

(b) $\sin x$

(c) $x^3 - 3x^2 + 1$

State also where the above functions are concave up, and where they are concave down.

2.4

When is the second derivative test inconclusive?

If $f'(x) = 0$ and $f''(x) < 0$, what does the second derivative tell you about x?

1 Solutions: Computing derivatives

1.1

Let f be a function. State the definition of the derivative of f at x.

Answer:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

1.2

Answers:

(a) $\cos(x)$

(b) $-\sin(x)$

(c) e^x

(d) $1/x$

(e) 1

1.3

Answers:

(a) $(x^3+9x^2+18)' = 3x^2+18x$

(b) $(9x\sin(x))' = (9x)'\sin(x) + 9x(\sin(x))' = 9\sin(x) + 9x\cos(x)$.

(c) $(\sin(e^x))' = \cos(e^x)\cdot(e^x)' = e^x\cos(e^x)$

(d) $(e^{\sin(x)})' = \cos(x)e^{\sin(x)}$

(e) $(\sin(x^3+3x+9))' = \cos(x^3+3x+9)\cdot(3x^2+3) = (3x^2+3)\cos(x^3+3x+9)$.

(f) $\left(\frac{\sin(x)}{x^2+1}\right)' = \frac{\cos(x)\cdot(x^2+1)-(2x)\sin(x)}{(x^2+1)^2} = \frac{(x^2+1)\cos(x)-2x\sin(x)}{(x^2+1)^2}$

(g) $(e^x\cdot\sin(x))' = (e^x)'\sin(x)+e^x(\sin(x))' = e^x\sin(x)+e^x\cos(x) = e^x(\sin(x)+\cos(x))$.

(h) $\left(\sqrt{\sqrt{e^x}}\right)' = \left(\left((e^x)^{1/2}\right)^{1/2}\right)' = \left(e^{x/4}\right)' = e^{x/4} \cdot 1/4 = \frac{1}{4}e^{x/4}$. You could also write this as $\frac{1}{4}\sqrt[4]{e^x}$.

(i) $(8^x)' = \left((e^{\ln 8})^x\right)' = \left(e^{x \ln 8}\right)' = e^{x \ln 8} \cdot \ln 8 = \ln 8 e^{x \ln 8}$.

2 Critical points and applications

2.1

(a) **x is a critical point of f if $f'(x) = 0$.**

(b) **x is an inflection point of f if f'' changes sign at x.**

2.2

Find the absolute minima and maxima of the following functions in the indicated intervals:

(a) $f(x) = x^2 + 3x + 1$ in the interval $[-1, 3]$

The only critical point is where $f'(x) = 2x + 3$ equals zero (i.e., when $x = -3/2$). This value of x is *not* inside the specified interval so we do not have to evaluate f at $x = -3/2$. Thus we are left to compare the values of f at the endpoints, -1 and 3. We find

$$f(-1) = -1, \qquad f(3) = 13.$$

Thus the absolute minimum is at $x = -1$, with value -1; the absolute maximum is at $x = 3$ with value 13.

(b) $f(x) = xe^x$ in the interval $[-8, 8]$.

We compute that $f'(x) = e^x + xe^x = (1 + x)e^x$. Because e^x never equals zero, f' is equal to zero only when $1 + x = 0$; that is, when $x = -1$. Thus we must evaluate the values of f at the endpoints and at $x = -1$:

$$f(-8) = -8e^{-8} = \frac{-8}{e^8}, \qquad f(-1) = -1e^{-1} = \frac{-1}{e}, \qquad f(8) = 8e^8.$$

Clearly the absolute maximum is at $x = 8$ with value $8e^8$. To find the absolute minimum, we must compare $f(-8)$ with $f(-1)$. Well, e^8 is a big number (it's bigger than 2^8, which is 256), so $8/e^8$ is less in absolute value than $8/256 = 1/32$. Thus we conclude that $-1/e < -8/e^8$. This means the absolute minimum is at $x = -1$ with value $\frac{-1}{e}$.

2.3

State where all the inflection points are of the following functions: State also where the above functions are concave up, and where they are concave down.

(a) $x^3 - 3x^2 + 1$

The first derivative is given by $3x^2 - 6x$. Thus the second derivative is $6x - 6$. $6x - 6$ equals 0 precisely when $x = 1$; moreover, the expression $6x-6$ changes sign when crossing $x = 1$. Thus $x = 1$ is the only inflection point. The sign o the second derivative is positive when $x > 1$, so the function is concave up when $x > 1$. Likewise, the function is concave down when $x < 1$.

(b) xe^x

The first derivative is given by $(x + 1)e^x$. the second derivative is given by $(x + 2)e^x$. This equals zero precisely when $x = -2$, and the second derivative clearly changes sign there (because $x + 2$ changes signs at $x = -2$, while e^x is always positive). Thus $x = -2$ is the only inflection point.

The function xe^x is concave up when $(x+2)e^x$ is positive—that is, when $x > -2$. It is concave down when $x < -2$.

(c) x^2e^x

The first derivative of this function is $(x^2+2x)e^x$. The second derivative is $(x^2 + 4x + 2)e^x$. By the quadratic formula, the polynomial $x^2 + 4x + 2$ has zeroes precisely when

$$x = \frac{-4 \pm \sqrt{4^2 - 2(1)(2)}}{2a} = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}.$$

Because $x^2 + 4x + 2$ is an upward pointing (x^2 has positive coefficient) parabola, we know that the sign of this function changes values at the zeroes. Thus, the inflection points are precisely $x = -2 - \sqrt{2}$ and $x = -2 + \sqrt{2}$. The function is concave down when the second derivative is zero, so when x is between the two inflection points. Otherwise, the function is concave up.

2.4

When is the second derivative test inconclusive? **The second derivative is inconclusive when the second derivative equals zero at a critical point.**

If $f'(x) = 0$ and $f''(x) < 0$, what does the second derivative tell you about x? **The second derivative tells us that x is a local maximum.**