Writing Assignment 10

Due Monday, November 2, 11:59 PM

Here is a definition that is notoriously difficult for calculus students:

Definition 0.0.2. Let $f(x)$ be a function, and choose a number a.

We say that $f(x)$ *has a limit at* a if the following holds:

There exists a number L such that, for every $\epsilon > 0$, there exists a $\delta > 0$ for which

$$\text{if } x \neq a \text{ and } |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

If $f(x)$ has a limit at a, we call L the limit of $f(x)$ at a.

This is a very confusing definition. For some of you, it may help to just examine the formal structure of the definition:

There exists a Lion such that, for any Elephant, there exists a Dog for which... (some condition on x determined by the Dog guarantees some relationship between the function at x, the Elephant, and the Lion.)

Anyhow, it's okay if you can't internalize this definition right away.

This week's prompt. Here's this week's writing assignment (which, as usual, is more like a thinking assignment—the writing should only come after hours of thinking):

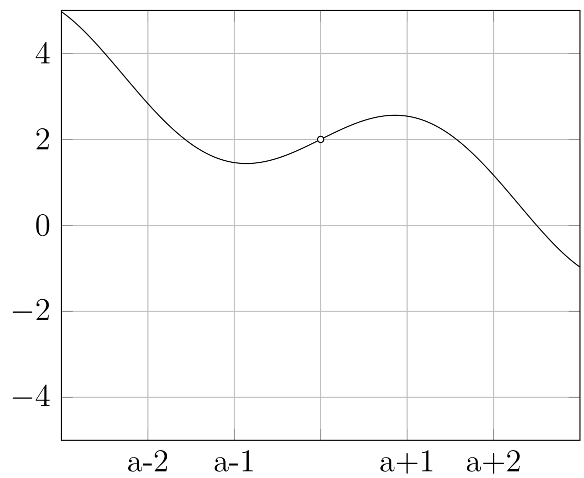
Does this definition fit your intuition of what a "limit" ought to be? How? How not? What's confusing about this definition? How can you make it less confusing?

I really want you to dig into these questions, and write what you come up with as you try to find answers. For many of you, I would strongly encourage you to hand-write your assignment so that you can draw pictures to illustrate your thoughts. (You can take a nice photo or scan your writing and upload to Canvas when you hand it in.)

The following pages give some further pictures and ideas to help you think about what's going on.

Exploring ϵ-δ visually

Below is the graph of a function $f(x)$, undefined at $x = a$.



Based on the graph, we suspect that

$$\lim_{x \to a} f(x) = 2.$$

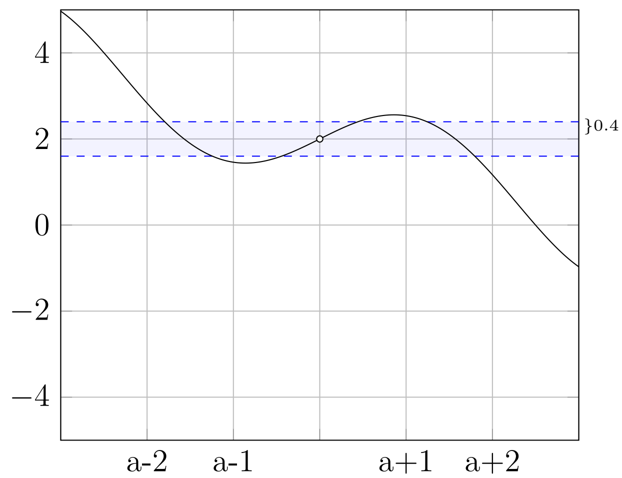
Exploratory questions. **(i)** Can we guarantee that so long as x is close enough to a, then $f(x)$ is within 0.4 units of the suspected limit? **(ii)** If so, how close does x have to be to a?

(a) On the graph above, draw the region of all points on the plane whose vertical coordinate is *strictly* between $2 - 0.4$ and $2 + 0.4$. (That is, between 1.6 and 2.4, non-inclusive.) Your answer should look like a horizontal strip.

(b) Does drawing this strip help you visualize the main questions?

As we will see, the take-away here is that you *can* guarantee to be within 0.4 of the suspected limit, so long as you choose x to be close enough to a.

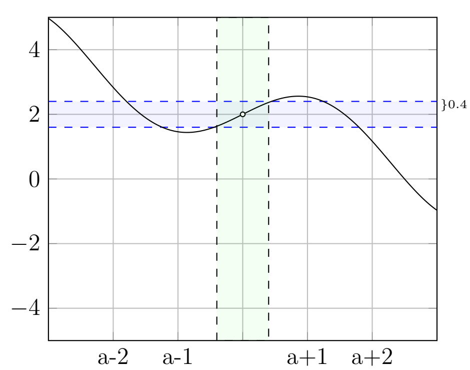
Recap

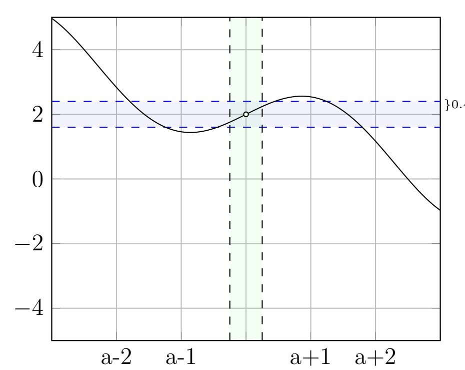


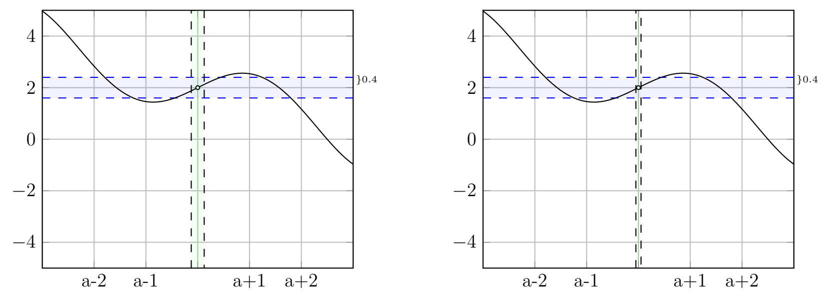
(a) Above, we have drawn the solution to (a) of the previous page. It is the strip between the line of height 2+0.4, and the line of height 2-0.4. Note that the edges of the strips are dashed, so that the vertical coordinates of the points in the strip are strictly between 1.6 and 2.4 (and not equal to either value).

(b) This helps us answer the main questions: So long as the graph of $f(x)$ is inside the strip, we know that $f(x)$ is within 0.4 of the suspected limit! (Remember that the suspected limit is 2.)

(c) Now, visually, we notice that in a region where x is close enough to a, the graph of $f(x)$ is always inside the strip. Here are sample examples:

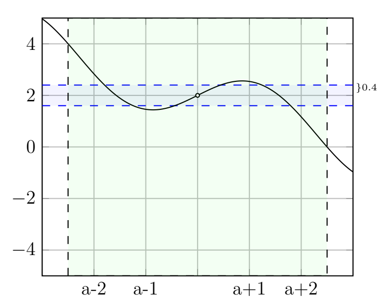




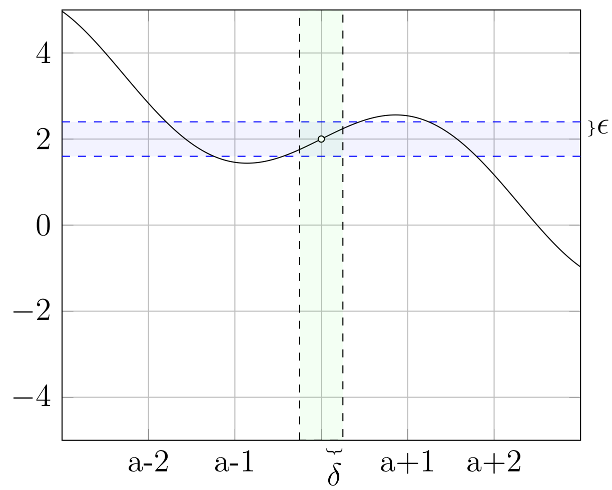


That is, *so long as x is in a thin-enough vertical strip, the graph of q in that vertical strip will also be in the horizontal strip.*

Warning. The "thin-enough" is important. If the vertical strip is too wide (that is, if we allow x to take values that are too far away from a) then $f(x)$ may escape the horizontal strip, meaning the value of $f(x)$ may be more than 0.4 away from the suspected limit. Below is an example where, because the vertical strip is too wide, the portion of $f(x)$ within the vertical strip is *not* contained in the horizontal strip.



Reading this will save you a lot of trouble. In ϵ-δ proofs, the vertical strips are always of width 2δ. The horizontal strips are always of height 2ϵ.



0.0.1 The ϵ-δ definition, in detail

Make sure you think carefully as you read this section.

Our mission is to understand the following statement:

$$\text{“}L \text{ is the limit of } f(x) \text{ as } x \text{ goes to } a.\text{”} \tag{0.0.1.1}$$

Informally, the above statement—(0.0.1.1)—can be rephrased as follows:

$$\text{“So long as } x \text{ is close enough to } a, \text{ we know } f(x) \text{ is close to } L.\text{”} \tag{0.0.1.2}$$

Now, we are going to interpret x being “close to a” as follows: that x is contained in some thin, vertical strip centered at a.

Likewise, we will interpret “$f(x)$ is close to L,” as “$f(x)$ is contained in some thin, horizontal strip centered at L.”

Let me stress again that the vertical strip is centered at a, and the horizontal strip is centered at height L.

In the drawings on the previous pages, we saw we could pictorially retranslate (0.0.1.2) to the following:

$$\begin{gathered}\text{“So long as the } x\text{-coordinate is contained in some thin-enough vertical strip,} \\ \text{we know } f(x) \text{ is contained in some thin horizontal strip.”}\end{gathered} \tag{0.0.1.3}$$

Now we will leap from the word “strip” to some algebraic notation. If we say that the vertical strip has width 2δ, then to say that the x-coordinate is contained in the vertical strip of 2δ centered at a is to say that

$$|x - a| < \delta.$$

Read that again if you didn't get it.

Likewise, to say that $f(x)$ is contained in a horizontal strip of height 2ϵ, centered at L, is to say that

$$|f(x) - L| < \epsilon.$$

Make sure you understand these inequalities.

Then, the statement (0.0.1.3) can finally be re-written as follows:

$$\text{“So long as } |x - a| < \delta, \text{ we know } |f(x) - L| < \epsilon.\text{”} \tag{0.0.1.4}$$

Make sure you understand how we got from (0.0.1.3) to (0.0.1.4).

So we see how ϵ, δ, and those confusing-looking inequalities show up.

But our definition also has a condition about $x \neq a$—this is just to emphasize that the limit a doesn't depend on the value of f *at* a, it only depends on the values of f at points *close to* a.

Let me put the cherry on top. The ϵ-δ definition of limit is equivalent to asserting the following: If $\lim_{x \to a} f(x) = L$, then you can always win a game.

What game? Your enemy dares you to fit the graph of $f(x)$ inside some strip of height 2ϵ. The only clue you are given is *which* ϵ your enemy chooses. You win if you can find a *width*, which we will call 2δ, so that whenever x is inside the vertical strip of that width, you know that the graph of $f(x)$ is within the horizontal strip with enemy-specified height.

0.0.2

Here is another interpretation. The letter ϵ is the old, Greek letter for e. The e stands for error.

You could think that $f(x)$ is the output of some machine—perhaps a machine that takes x kilograms of ore and turns it into $f(x)$ kilograms of useful iron.

Let's say you want to produce about L kilograms of useful iron. Any process has error or uncertainty, so getting exactly L is hard, but you certainly don't want to be too far away from L. In fact, your client will be angry if they don't get at least $L - \epsilon$ kilograms. And certainly, you don't want to give away more iron than you need to, so you want to produce at most $L + \epsilon$ kilograms of iron. That is, you want to produce somewhere between $L + \epsilon$ and $L - \epsilon$ kilograms of useful iron.

But measuring how much ore you put into your machine is always a delicate issue—it's hard to put in *exactly* the right amount of ore. Hmm. How accurate do you need the *input* to be to make sure you produce the desired amount of iron?

δ is the measure of your needed accuracy. If a kilograms of ore outputs exactly L kilograms of iron, a tiny mistake in measuring x kilograms—that is, a small enough mistake that you actually put in somewhere between $a - \delta$ and $a + \delta$ kilograms—should guarantee that you output between $L - \epsilon$ and $L + \epsilon$ kilograms of useful iron. You just need to know how small "small enough" actually is! That is, how small does δ need to be once you're aiming for ϵ error?

In sum: Given the tolerance ϵ, you want to find the permitted inaccuracy δ to please your client and not be wasteful.

Now, when you can find a δ given any ϵ, your machine is a great one. Mathematically, this greatness translates to "$f(x)$ has a limit."

But you might have a machine that is completely unpredictable and unreliable for certain amounts of input ore. It just gets downright finicky when $a = 10$. And for some values of ϵ, no matter how small you can reduce your inaccuracy δ, you just cannot guarantee an output within the error tolerance of ϵ. That's an unfortunately bad machine. This mathematically translates into "$f(x)$ does not have a limit at 10."