Lecture 32

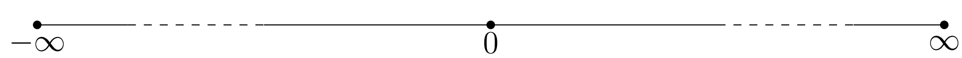
Limits equaling ∞

Warning 32.0.1. We will be using the symbol ∞ a lot. This symbol stands for "infinity." I want you to know that the way we use ∞ in calculus class can be detrimental to understanding the other uses of infinity in mathematics.

There are "infinitely many" integers; this notion of infinity answers the question "how many?". The "how many" notion is subtly, but definitely, different from the notion of ∞ that we'll use in calculus, which is more dynamical—our notion answers the question "are our numbers eventually getting bigger and bigger?"

32.0.1 Infinity, in our class

Where are ∞ and $-\infty$? This is controversial among some calculus instructors; but in my class, we will somtimes treat ∞ and $-\infty$ as though they are "numbers." In fact, you should imagine that I've added two ends to the number line:



So for example, between 0 and ∞ lies *every* positive real number. Between $-\infty$ and 0 lies every negative real number. ∞ is larger than any number; $-\infty$ is lesser than any number.

Remark 32.0.2. This should give you some idea for what it means to *approach* infinity. It means that, for any point T on the real line, you eventually surpass and stay larger than T. Likewise, for you to approach $-\infty$ means that, for any number T on the real line, you eventually become more negative than, and stay more negative than, T.

32.0.2 Arithmetic with ∞ and $-\infty$

Of course, you should be able to add/subtract/multiply/divide numbers. Here are the basic rules you need to remember; they are what you would have guessed. (Below, remember that ∞ and $-\infty$ are *NOT* real numbers.)

- Addition and multiplication are still commutative.

Here are the rules involving addition and subtraction:

- If x is a real number, $x + \infty = \infty$ and $x + (-\infty) = (-\infty)$.[1]
- $\infty + \infty = \infty$ and $(-\infty) + (-\infty) = (-\infty)$.

When taking products and quotients, we must never involve zero with $\pm\infty$:

- If x is a *positive* real number, $x \times \infty = \infty$ and $x \times (-\infty) = (-\infty)$.
- If x is a *negative* real number, $x \times \infty = -\infty$ and $x \times (-\infty) = \infty$.
- If x is a *positive* real number, $\infty/x = \infty$ and $(-\infty)/x = (-\infty)$.
- If x is a *negative* real number, $\infty/x = -\infty$ and $(-\infty)/x = \infty$.
- If x is a real number with $x \neq 0$, then $x/\infty = 0$ and $x/(-\infty) = 0$.
- $\infty \times \infty = \infty$ and $\infty \times (-\infty) = (-\infty)$ and $(-\infty) \times (-\infty) = \infty$.

Undefined expressions: Finally, just as you cannot divide a real number by zero, there are certain operations that are undefined when involving $\pm\infty$:

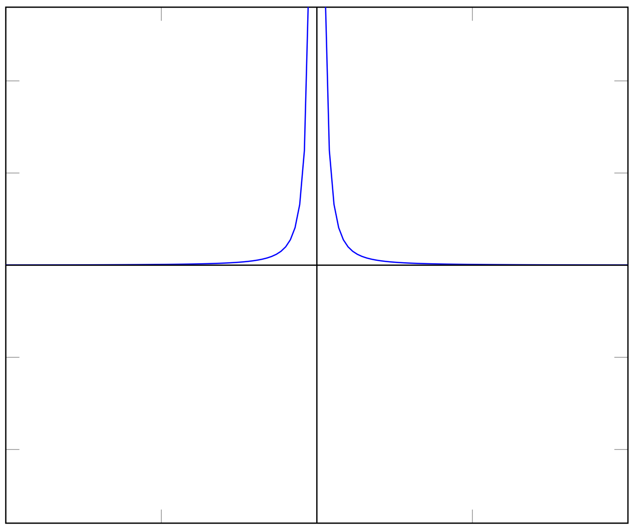
- ∞ - ∞ and $-\infty + \infty$ are undefined.
- ∞/∞ and $-\infty/\infty$ and $\infty/(-\infty)$ and $(-\infty)/(-\infty)$ are all undefined.
- $0 \times \infty$ and $0 \times (-\infty)$ are undefined.
- $0/\infty$ and $0/(-\infty)$ and $\infty/0$ and $(-\infty)/0$ are undefined.

[1] In particular, $\infty - x = \infty$ and $(-\infty) - x = (-\infty)$.

32.0.3 Limits equaling infinity

We'll talk about limits equaling ∞ via examples.

Example 32.0.3. Consider the function $f(x) = 1/x^2$. Here's a graph of it:



As you know, $f(x) = 1/x^2$ is *not* defined at $x = 0$. However, does f seem to "want" to do something as x approaches zero?

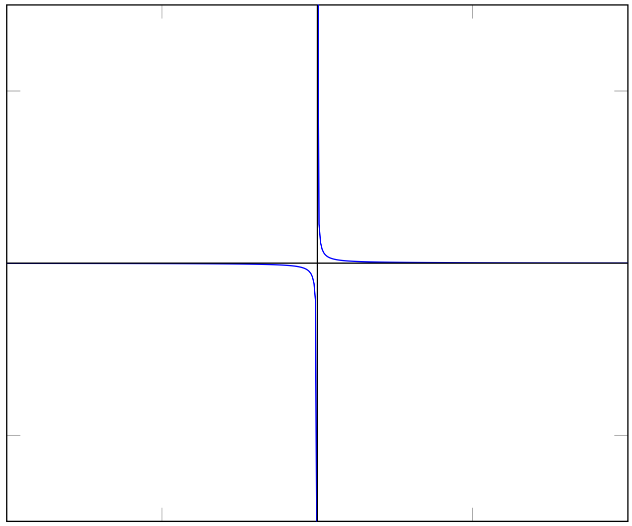
As you see from the graph, f is "spiking" at $x = 0$, and becoming larger and larger. In fact, if there's a height H that you want to surpass, all you have to do is make sure that x is small enough. For every small-enough x, we know $f(x)$ will be larger than H.

Thus, we say:

$$\lim_{x \to 0} f(x) = \infty.$$

This is our first use of ∞ in calculus class!

Example 32.0.4. We can talk about left and right limits equaling ∞, too. Consider the function $f(x) = 1/x$. Here's a graph of it:



As you can see, as we approach the origin from the *right*, the graph of f is spiking upward again. We can talk about this righthand limit:

$$\lim_{x\to 0^+} f(x) = \infty.$$

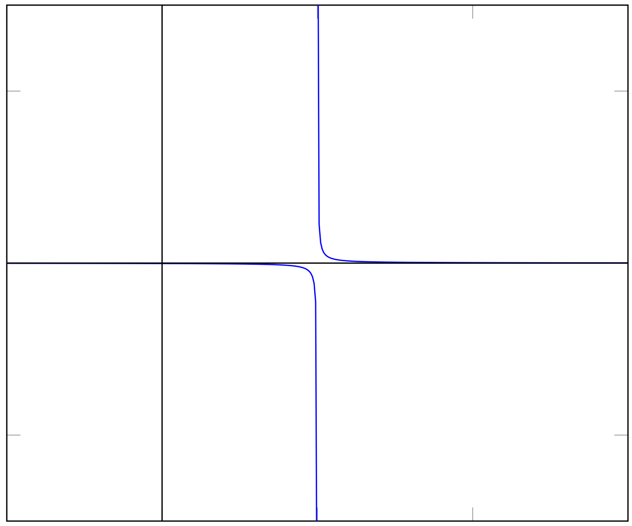
However, as we approach $x = 0$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

$$\lim_{x\to 0^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x\to 0} f(x) \text{ does not exist.}$$

Example 32.0.5. Consider the function $f(x) = 1/(x - 0.2)$. Here's a graph of it:



As you can see, as we approach 0.2 from the *right*, the graph of f is spiking upward again. So

$$\lim_{x \to 0.2^+} f(x) = \infty.$$

However, as we approach $x = 0.2$ from the left, the graph of f is spiking *downward*, and f is approaching $-\infty$. Thus, we say:

$$\lim_{x \to 0.2^-} f(x) = -\infty.$$

Note that the lefthand limit and the righthand limit do *not* agree. Just like limits for real numbers (and not $\pm\infty$), because the two one-sided limits do not agree, we can say:

$$\lim_{x \to 0.2} f(x) \text{ does not exist.}$$

There is nothing special about 0.2. In fact, for any real number C, we have that

$$\lim_{x \to C^+} \frac{1}{x - C} = \infty, \qquad \lim_{x \to C^-} \frac{1}{x - C} = -\infty, \qquad \lim_{x \to C} \frac{1}{x - C} = \text{does not exist.}$$

32.1 Limit rules, revisited (this time with ∞)

Once you know how to add/multiply/divide/subtract with ∞, and once you know the basic limits, you can begin to compute limits of more complicated functions.

Here are the basic limit laws for infinity; they are like the old ones, just with more caveats about being careful:

1. (**New:** Limits of $1/(x-C)$). For any real number C, we have that
$$\lim_{x\to C^-}\frac{1}{x-C}=-\infty, \qquad \text{and} \qquad \lim_{x\to C^+}\frac{1}{x-C}=\infty.$$
(Make sure to take a look at Example 32.0.5 if you haven't yet.)

2. (Scaling law). When the righthand side is defined, for any real number m, we have
$$\lim_{x\to a} mf(x)=m\lim_{x\to a}f(x).$$
New point of caution: The righthand side is undefined if $m=0$ and if $\lim_{x\to a}f(x)=\pm\infty$.

3. (Puncture law). If $f(x)=g(x)$ away from a, then
$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x).$$

4. (Product law) We have that
$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x).$$
New point of caution: Importantly, the righthand side is not defined when multiplication is not defined—for example, $0\cdot\infty$ is undefined for us. When the righthand side is undefined, you have to *try something different* from the product rule to determine the limit.

5. (Quotient law) We have that
$$\lim_{x\to a}(\frac{f(x)}{g(x)})=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}.$$
New point of caution: Importantly, the righthand side is not defined when division is not defined—for example, $0/\infty$ is undefined. When the righthand side is undefined, you have to *try something different* from the quotient rule to determine the limit.

Remark 32.1.1. Limit laws also work for one-sided limits! This is a good thing. For example,
$$\lim_{x\to a^+}(f(x)\cdot g(x))=\lim_{x\to a^+}f(x)\cdot\lim_{x\to a^+}g(x).$$

Example 32.1.2 (This is an example you should memorize the result of). Let's try to establish that

$$\lim_{x\to 0} \frac{1}{x^2} = 0.$$

It suffices to compute both one-sided limits, and to show that they are the same. Here's one:

$$\lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^+} (\frac{1}{x}\cdot\frac{1}{x}) \tag{32.1.1}$$
$$= \lim_{x\to 0^+} \frac{1}{x} \cdot \lim_{x\to 0^+} \frac{1}{x} \tag{32.1.2}$$
$$= \infty\cdot\infty \tag{32.1.3}$$
$$= \infty. \tag{32.1.4}$$

The first line is just algebra. The next line is using the product rule for one-sided limits. Then we are using the fact that we know already the one-sided limits for $1/x$. The last line follows from our rules about arithmetic with ∞.

And here's the other one-sided limit:

$$\lim_{x\to 0^-} \frac{1}{x^2} = \lim_{x\to 0^-} (\frac{1}{x}\cdot\frac{1}{x}) \tag{32.1.5}$$
$$= \lim_{x\to 0^-} \frac{1}{x} \cdot \lim_{x\to 0^-} \frac{1}{x} \tag{32.1.6}$$
$$= (-\infty)\cdot(-\infty) \tag{32.1.7}$$
$$= \infty. \tag{32.1.8}$$

In sum, we see that both one-sided limits agree, so $1/x^2$ has a limit at 0. We can conclude:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Example 32.1.3. Let's compute

$$\lim_{x\to 0^+} \frac{1+x}{4x^2}.$$

The fastest approach is to use the product law:

$$\lim_{x\to 0^+} \frac{1+x}{4x^2} = \lim_{x\to 0^+} \frac{1+x}{1} \cdot \lim_{x\to 0^+} \frac{1}{4x^2} \tag{32.1.9}$$
$$= 1 \cdot \lim_{x\to 0^+} \frac{1}{4x^2} \tag{32.1.10}$$
$$= 1 \cdot \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x^2} \tag{32.1.11}$$
$$= \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x^2} \tag{32.1.12}$$
$$= \frac{1}{4}\infty \tag{32.1.13}$$
$$= \infty. \tag{32.1.14}$$

The first line is the product law. (32.1.11) is computing the limit of $\frac{1+x}{1}$. (This is something you already knew how to do.) (32.1.12) is the scaling law. The next line is algebra. (32.1.13) follows from our knowledge of the limit of $1/x^2$ at 0. The last line is arithmetic using ∞.

Here is a different, very tedious approach:

$$\lim_{x\to 0^+} \frac{1+x}{4x^2} = \lim_{x\to 0^+} \left(\frac{1}{4x^2} + \frac{x}{4x^2}\right) \tag{32.1.15}$$
$$= \lim_{x\to 0^+} \frac{1}{4x^2} + \lim_{x\to 0^+} \frac{x}{4x^2} \tag{32.1.16}$$
$$= 4 \lim_{x\to 0^+} \frac{1}{x^2} + \lim_{x\to 0^+} \frac{x}{4x^2} \tag{32.1.17}$$
$$= 4 \cdot \infty + \lim_{x\to 0^+} \frac{x}{4x^2} \tag{32.1.18}$$
$$= \infty + \lim_{x\to 0^+} \frac{x}{4x^2} \tag{32.1.19}$$
$$= \infty + \lim_{x\to 0^+} \frac{1}{4x} \tag{32.1.20}$$
$$= \infty + \frac{1}{4} \lim_{x\to 0^+} \frac{1}{x} \tag{32.1.21}$$
$$= \infty + \frac{1}{4} \cdot \infty \tag{32.1.22}$$
$$= \infty + \infty \tag{32.1.23}$$
$$= \infty. \tag{32.1.24}$$

The first line is algebra, and the next line is the addition rule. Note that we don't know whether the sum will be well-defined[2] at this stage, but we proceed crossing our fingers. Then I kept simplifying the lefthand term in the summation, knowing that $\lim 1/x^2 = \infty$ and using the scaling law. Line (32.1.20) follows from the puncture law. Then I use the scaling law, and then my knowledge of $\lim_{x\to 0^+} \frac{1}{x}$. The last few lines are following the arithmetic of ∞.

Example 32.1.4. Let's compute

$$\lim_{x\to 3^+} \frac{5x}{x-3}.$$

We have the following string of equalities:

$$\lim_{x\to 3^+} \frac{5x}{x-3} = \lim_{x\to 3^+} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3} \tag{32.1.25}$$

$$= 15 \cdot \lim_{x\to 3^+} \frac{1}{x-3} \tag{32.1.26}$$

$$= 15 \cdot \infty \tag{32.1.27}$$

$$= \infty \tag{32.1.28}$$

The first equality is the product law for limits—note that we did not know[3] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (32.1.26) is evaluating the limit for $5x$, which we knew how to do already. (32.1.27) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞.

Example 32.1.5. Let's compute

$$\lim_{x\to 3^-} \frac{5x}{x-3}.$$

[2]For example, if at the end we find a sum of the form $\infty - \infty$, we are at a loss—this expression is not defined.

[3]We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^+} \frac{1}{x-3}$ would yield something non-sensical like $0 \cdot \infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

We have the following string of equalities:

$$\lim_{x\to 3^-} \frac{5x}{x-3} = \lim_{x\to 3^-} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3} \tag{32.1.29}$$
$$= 15 \cdot \lim_{x\to 3^-} \frac{1}{x-3} \tag{32.1.30}$$
$$= 15 \cdot -\infty \tag{32.1.31}$$
$$= -\infty \tag{32.1.32}$$

The first equality is the product law for limits—note that we did not know[4] that we are allows to use it until the second-to-last line, but we tried computing it anyway (and got lucky that it worked!). (32.1.30) is evaluating the limit for $5x$, which we knew how to do already. (32.1.31) is using our knew limit law for functions of the form $1/(x-C)$. Note that C *is* also where we're taking the limit—this is an important part of the law.

The last equality is using the arithmetic rules for ∞.

Exercise 32.1.6. Compute—using the limit laws above—the one-sided limits

$$\lim_{x\to 3^-} \frac{\ln x}{x-3} \qquad \text{and} \qquad \lim_{x\to 3^+} \frac{\ln x}{x-3}.$$

Does $\lim_{x\to 3} \frac{\ln x}{x-3}$. exist?

Exercise 32.1.7. Compute

$$\lim_{x\to 0^+} \frac{1}{e^x-1} \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{e^x-1}.$$

Try to think this through *without* using the limit laws above—they won't help.

Does $\lim_{x\to 0} \frac{1}{e^x-1}$ exist?

I want to very carefully walk through this last exercise. How would we compute

$$\lim_{x\to 0^+} \frac{1}{e^x-1}?$$

Clearly the denominator is causing us trouble. The quotient law doesn't apply because the limit of the denominator is 0.

[4]We did not know we could use it because we did not know whether the product $\lim_{x\to 3} 5x \cdot \lim_{x\to 3^-} \frac{1}{x-3}$ would yield something non-sensical like $0 \cdot \infty$ upon simplification. When the product *is* sensible, we can safely rely on the product law.

However, let's think about what's happening to $e^x - 1$ as x approaches 0 from the right. When $x > 0$, we know that $e^x > e^0$. In other words, $e^x > 1$.

Thus, as x approaches 0 from the right, the denominator remains *positive*, but shrinks to zero. (As x approaches 0 from the right, e^x shrinks in size, and e^x becomes closer and closer to 1 while remaining larger than 1. As a result, $e^x - 1$ becomes closer and closer to 0 while remaining positive.)[5]

So $\frac{1}{e^x-1}$, as we shrink x to 0 from the right, is positive, and growing larger and larger (because we are dividing 1 by smaller and smaller numbers). This intuition suggests

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = \infty.$$

Likewise, as x approaches 0 from the left, e^x is less than 1, but is growing in size to 1. Thus $e^x - 1$ is negative, but approaching 0. So we conclude

$$\lim_{x\to 0^+} \frac{1}{e^x - 1} = -\infty.$$

We will see how to compute this more rigorously next time, when we also talk about limits *at* $x = \pm\infty$.

32.2 For next time

I expect you to be able to compute limits similar to the following:

$$\lim_{x\to 0^+} \frac{1+x}{4x^2}.$$

$$\lim_{x\to 3^+} \frac{5x}{x-3}.$$

$$\lim_{x\to 0} \frac{1}{x^2}$$

$$\lim_{x\to C^-} \frac{1}{x-C} \qquad \text{and} \qquad \lim_{x\to C^+} \frac{1}{x-C} \qquad \text{and} \qquad \lim_{x\to C} \frac{1}{x-C}.$$

[5]Make sure you understand this!