Lecture 31

Continuity, intermediate value theorem, and puncture law

31.1 More on continuity

You have entered the journey of mathematical maturity that everybody has to go through: You're being given abstract definitions, but you don't still understand what they mean in concrete situations, nor why they're useful.

31.2 Practice with the straightforward limit laws

Here are some limit laws we didn't have time to practice last time. Get in your groups and try them out.

Exercise 31.2.1. Using the limit laws, convince yourself that if $h(x) = x^2$, then

$$\lim_{x \to a} h(x) = h(a).$$

(Hint: Use the functions $f(x) = x$ and $g(x) = x$, along with the product law.)

Exercise 31.2.2. Using the limit laws, show that **limits subtract**.

More precisely, if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then so does $\lim_{x \to a} (f(x) - g(x))$. Moreover,

$$\lim_{x \to a} (f(x) - g(x)) = \left(\lim_{x \to a} f(x)\right) - \left(\lim_{x \to a} g(x)\right)$$

(Hint: Use the fact that limits scale, taking your scaling constant to be $m = -1$, and use the fact that limits add.)

Exercise 31.2.3. Use the limit laws to compute

$$\lim_{x\to 1}\left(\frac{x^2+3}{x}\right).$$

What goes wrong when you try to compute the limit as $x \to 0$?

31.3 Composition law

There is another powerful way to make new functions out of old: Composition. Limits respect composition, too, so long as the outermost function is continuous at the limit of the innermost function:

Composition law. Let $g(x)$ and $f(x)$ be functions, and suppose you know that $f(x)$ is continuous at $\lim_{x\to a} g(x)$. Then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Informally, this means you can "move the limit inside" of f so long as f is continuous where it counts.

Exercise 31.3.1. Using the composition law, and your knowledge that $f(x) = x^2$ is continuous at every point[1], compute

$$\lim_{x\to 3} f(g(x))$$

if g is a function for which $\lim_{x\to 3} g(x) = \pi$.

Warning 31.3.2. To use the composition law, the "outermost" function needs to be continuous where it counts. (Re-read the composition law if this wasn't clear when you first read it!)

31.4 One-sided limits

Sometimes, a function approaches a value from the right; sometimes, the function approaches a value from the left. These values might be different!

[1] You proved this in Exercise 31.2.1!

Definition 31.4.1 (One-sided limits)**.** If $f(x)$ wants to converge to a value as x approaches a from the right, we call this value the *righthand limit* of $f(x)$ at a, and we denote this value by

$$\lim_{x \to a^+} f(x).$$

(Note the plus sign on the a.)

If $f(x)$ wants to converge to a value as x approaches a from the left, we call this value the *lefthand limit* of $f(x)$ at a, and we denote this value by

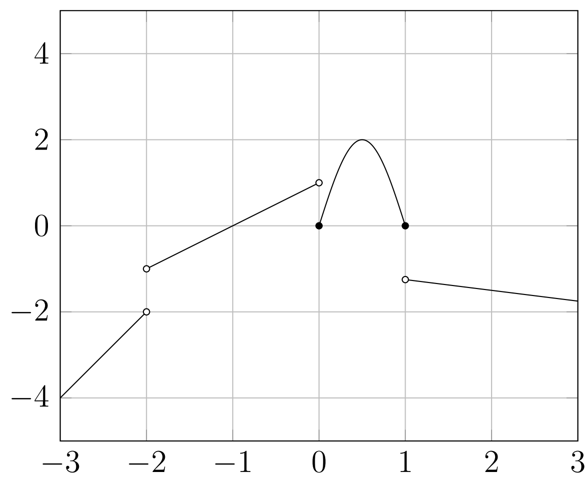
$$\lim_{x \to a^-} f(x).$$

(Note the *minus* sign on the a.)

A lefthand limit or a righthand limit is called a *one-sided limit*.

Warning 31.4.2. Just like limits, a one-sided limit may not exist!

Exercise 31.4.3. Below is the graph of a function $f(x)$.



Based on the graph, give your best guest for the following one-sided limits.

(a) $\lim_{x \to -2^-} f(x)$.

(b) $\lim_{x \to -2^+} f(x)$.

(c) $\lim_{x \to 1^+} f(x)$.

(d) $\lim_{x \to 1^-} f(x)$.

Exercise 31.4.4. Consider the function

$$f(x) = \begin{cases} 0 & x > 0 \text{ and } x \text{ is irrational} \\ 1 & x > 0 \text{ and } x \text{ is rational} \\ 13 & x < 0. \end{cases}$$

Tell me whether $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ exist, and if they exist, what their values are.

31.5 Using one-sided limits

Here is our first theorem. A *theorem* is a true statement that requires an involved proof, and the true statement is so useful that we should[2] know it for future use.

Theorem 31.5.1. The following statements are equivalent:

1. $f(x)$ has a limit at a.
2. Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, and the one-sided limits agree.

Moreover, in this situation, we can conclude that

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Remark 31.5.2. The term "equivalent" has a precise meaning here. It means that "if the first statement is true, then the second statement true," *and* that "if the second statement is true, then the first statement is true."

In other words, if f has a limit at a, then it has both one-sided limits there, and they agree. Conversely, if f has both one-sided limits at a and they agree, then f has a limit at a.

Example 31.5.3. Somebody tells you the following information:

$$\lim_{x\to 1^+} f(x) = 3 \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 10.$$

Then you know that $\lim_{x\to 1} f(x)$ does not exist, because the two one-sided limits do not agree.

[2]That means you'll be tested on it!

Example 31.5.4. Somebody tells you the following information:

$$\lim_{x\to 2^+} f(x) = 10 \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = 10.$$

Then you know that $f(x)$ *does* have a limit at 2, because the two one-sided limits agree (that is, they have the same value). Moreover, you can conclude that

$$\lim_{x\to 2} f(x) = 10.$$

31.6 Summary of straightforward limit laws

Limits of constants. If $f(x)$ is a constant function[3] with value C, then

$$\lim_{x\to a} f(x) = C$$

regardless of a.

Limits of x. For the function $f(x) = x$, we have that

$$\lim_{x\to a} f(x) = a.$$

Warning 31.6.1. In the following limit laws, you must *already know* that all the limits on the righthand side of the equality exist before being able to conclude the existence of, and compute, the limit on the lefthand side.

Limits scale.

$$\lim_{x\to a} (m \cdot f(x)) = m \cdot \left(\lim_{x\to a} f(x)\right)$$

Limits add.

$$\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

Limits multiply.

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

Limits divide.

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

so long as $\lim_{x\to a} g(x) \neq 0$.

[3]This means $f(x) = C$ for some number C. Put another way, the graph of $f(x)$ is just a flat, horizontal line.

31.7 Continuity

Definition 31.7.1. A function $f(x)$ is called *continuous* if it is continuous at every point that $f(x)$ is defined.

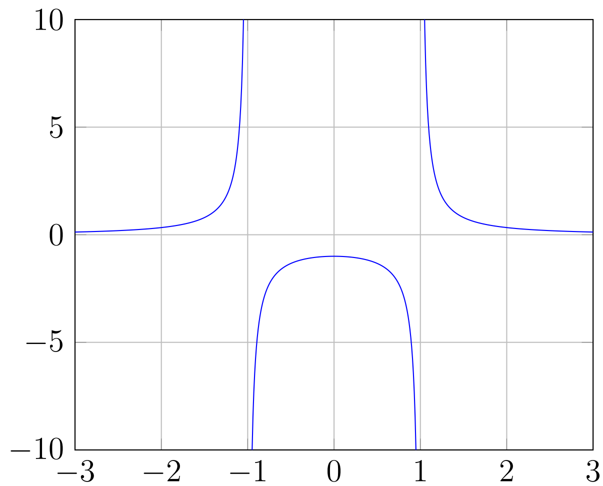
Let me give the following non-mathematical, but very helpful, intuition:

Intuition: "*A continuous function is one for which you can draw the graph of the function without ever having to lift your pencil from the paper.*"

Warning 31.7.2. This intuition fails in small ways. For example, suppose that

$$f(x) = \frac{1}{(x+1)(x-1)}.$$

Here is the graph of $f(x)$:



You can see that f is not defined at $x = 1$ and $x = -1$. So there is no way that you can draw the whole graph without lifting your pencil. But f is still a continuous function, because the value of f agrees with the limit of f at every point f is defined.

Regardless, "never have to lift your pencil" is a useful way to think about what continuity looks like. This agrees with another intuition: A continuous function has no "sudden jumps."

Example 31.7.3. As it turns out, almost every function with a "formula" that you know is continuous. Here is a list of some examples of continuous functions:

1. $f(x) = 10$ (and all other constant functions)

2. $f(x) = x$ (and all other linear functions)
3. $f(x) = 3x^3 + 4x^2 + 9$ (and all other polynomials—you can actually prove this based on the basic limit laws from last lecture)
4. $f(x) = \frac{3x^2+1}{x-3}$ (and all other functions that are quotients of polynomials—you can actually prove this based on the basic limit laws from last lecture)
5. $f(x) = |x|$ (I bet you can prove this function is continuous!)
6. $f(x) = \sin(x)$ (and all other trig functions)
7. $f(x) = \sqrt{x}$
8. $f(x) = x^p$, for any real number p. (You should be familiar with the special cases when p is a negative integer like $p = -1$ or $p = -2$, and when p is a fraction like $p = 1/3$ or $p = 2/3$.)
9. $f(x) = e^x$
10. $f(x) = \ln(x)$

The continuity of the last five examples require some proofs that we won't go over in this class.

From now on, you may use—and are expected to know—that all the functions above are continuous.

Example 31.7.4. You have now been told that $x \mapsto x^{1/n}$ is continuous. We can use the composition law to deduce the following **Root Law**: The root of the limit is the limit of the root.

That is, prove that if $\lim_{x\to a} f(x)$ exists,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Here is the proof. Let $h(x) = x^{1/n}$. Then *because $h(x)$ is a continuous function,* so we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{31.7.1}$$

(Line (31.7.1) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a} (f(x))^{1/n} \qquad , \qquad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{31.7.2}$$

Stringing (31.7.1) and (31.7.2) together, we find:

$$\lim_{x\to a}(f(x)^{1/n}) = \left(\lim_{x\to a} f(x)\right)^{1/n}. \tag{31.7.3}$$

And now let's just remember that raising something to the $1/n$ power is the same thing as taking the nth root. So (31.7.3) becomes

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

And we're done!

Warning 31.7.5. The root law only makes sense when taking nth roots makes sense. For example, if n is even, then the law only makes sense if $\lim_{x\to a} f(x)$ is not negative.

Example 31.7.6. You have now been told that $x \mapsto x^p$ is continuous. We can use the composition law to deduce the following **Power Law**: The power of the limit is the limit of the power.

That is, prove that if $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a} (f(x)^p) = \left(\lim_{x\to a} f(x)\right)^p.$$

Here is the proof. Let $h(x) = x^p$. Then *because $h(x)$ is a continuous function,* we can use the composition law to conclude that

$$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x)). \tag{31.7.4}$$

(Line (31.7.4) is where we are using the composition law.) Now let's just plug in what $h(x)$ is to simplify both sides:

$$\lim_{x\to a} h(f(x)) = \lim_{x\to a}(f(x))^p \qquad , \qquad h(\lim_{x\to a} f(x)) = \left(\lim_{x\to a} f(x)\right)^p. \tag{31.7.5}$$

Stringing (31.7.4) and (31.7.5) together, we find:

$$\lim_{x\to a}(f(x))^p = \left(\lim_{x\to a} f(x)\right)^p.$$

That's the power law we wanted to prove, so our proof is complete!

Warning 31.7.7. The power law only makes sense when taking pth powers makes sense. For example, if p is negative, then the law only makes sense if $\lim_{x\to a} f(x)$ is not zero.

31.8 The Intermediate Value Theorem

31.8.1 Some warm-up exercises

Exercise 31.8.1. Consider the function $f(x) = x^2 + 10$. Does this function have a root?

(Recall that a *root* is a value of x for which $f(x)$ equals zero. So, another way to rephrase the question: is there a value of x such that $x^2 + 10$ equals zero?)

Explain.

Exercise 31.8.2. Consider the polynomial function $f(x) = x^5 + 7x^4 - 22x + 19$. (This function is complicated, I know!)

Let me tell you that $f(-10)$ has the value -29,761. Also, $f(3)$ equals 763.

Based on this information, does $f(x)$ have a root?

(This question is *not* asking you to *find* a root; it's asking you whether a root *exists*.)

Explain. Can you explain in such a way where you can ignore/forget how complicated $f(x)$ looks?

We didn't get to go over the word "theorem" in the previous class (though it was used on the hand-out). A *theorem* is a mathematical fact that is very useful, and that somebody proved for your use. Because somebody has proven our theorems to be true[4], you may utilize theorems whenever you like in the future.

Here is a theorem.

Theorem 31.8.3 (Intermediate Value Theorem)**.** Let $f(x)$ be a continuous function, and choose two real numbers a and b with $a < b$. [5] Then for any number N between $f(a)$ and $f(b)$, [6] there is a number c between a and b so that $f(c) = N$.

Put another way, on the way from a to b, the graph of f attains (at least) every height between $f(a)$ and $f(b)$.

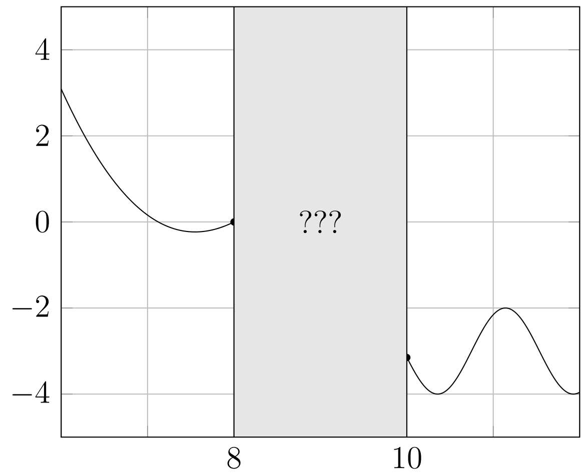
Remark 31.8.4. Sometimes, we abbreviate the Intermediate Value Theorem by "IVT" (especially when we are running out of time on exams or quizzes).

Example 31.8.5. Here is a graph of a function $f(x)$ that your friend began to make, then stopped part-way:

[4]The beauty is, if you want, you can prove it too! It just won't be easy with the tools you've learned so far, but you can do it.

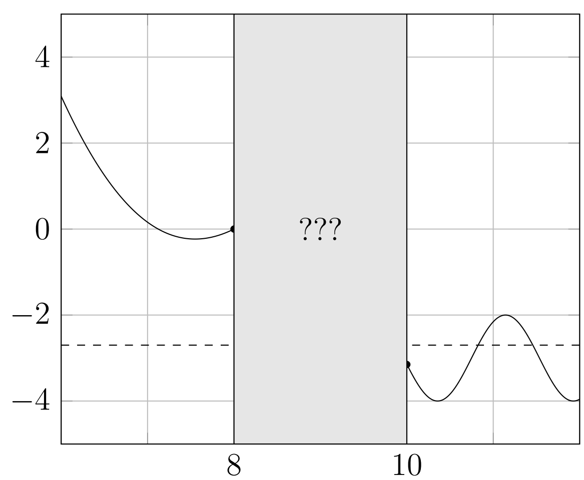
[5]You should imagine these numbers to be on the x-axis.

[6]You should imagine N, $f(a)$, and $f(b)$ to be on the y-axis



So you have no idea what $f(x)$ looks like in the region between 8 and 10. However, you do know that $f(8) = 0$ and $f(10) = -3$. Therefore, *if* $f(x)$ *is continuous*, then the Intermediate Value Theorem tells you that $f(x)$ must hit (at least) every number between 0 and -3, at least once.[7]

For example, -2.7 is a number between 0 and -3. So, though you *do not know where*, you do know that $f(x)$ must equal -2.7 at *some value* of x between 8 and 10. [8] Here is a pictorial way to think about it:



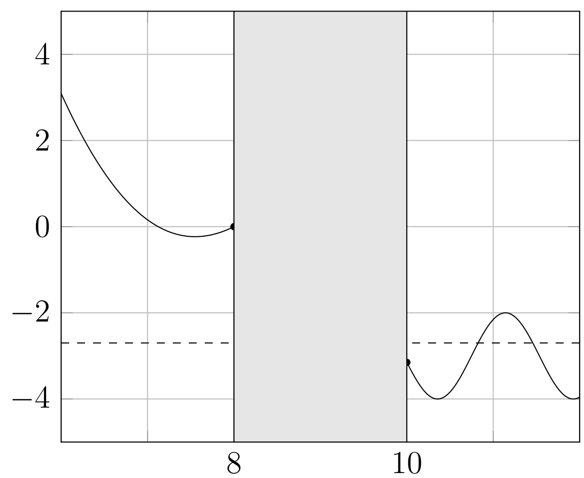
[7] In this example, $a = 8$ and $b = 10$.

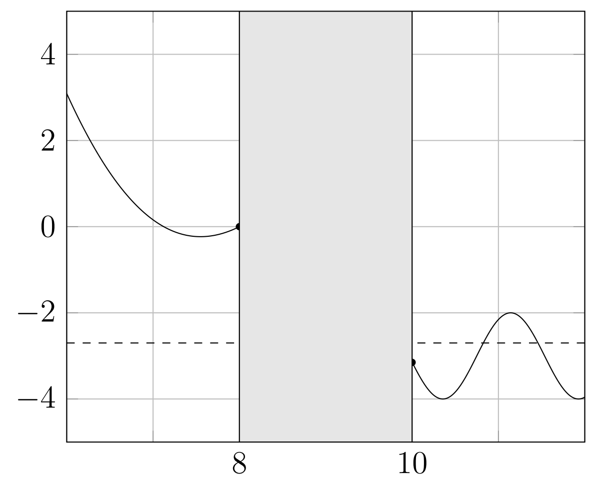
[8] In terms of the letters used in Theorem 31.8.3, $N = -2.7$. And c is the *some value* between 8 and 10.

We have drawn, in dashes, the line at height -2.7. Because $f(x)$ is continuous, to get from height 0 to height -3, the graph of $f(x)$ *must* cross over this line at some point in the grey region. We don't know where $f(x)$ crosses the line, but it does so *somewhere* between $x = 8$ and $x = 10$.

Remark 31.8.6. Note that, in Example 31.8.5, the graph of $f(x)$ crosses over the line of height -2.7 *outside* the grey region as well. That's all well and good, but the intermediate value theorem only guarantees something about the *grey region*—i.e., about the region between a and b.

Remark 31.8.7. Here are some examples of continuous functions that could fill in the grey region from Example 31.8.5:





Note that $f(x)$ may attain N at *more than one value of* c. (You can see this graphically in the lefthand example: The graph of $f(x)$ crosses the horizontal line of height $N = -2.7$ three times.)

Note that $f(x)$ *does not need to stay inbetween* $f(a)$ *and* $f(b)$. (You can see this on the righthand example.) That is, even if $a < c < b$, it need *not* be true that $f(c)$ is between $f(a)$ and $f(b)$.

Exercise 31.8.8. Do Exercise 31.8.2 again, using the IVT. Make sure you know what the values of a, b, and N are.

Do you know the value of c?

31.9 Intermediate value theorem on a closed interval

Recall that a *closed* interval is an interval of the form

$$[a, b]$$

with $a < b$. For example, $[2, 7]$ is the interval of all numbers between 2 and 7, *including* 2 and 7.

An *open* interval is an interval of the form

$$(a, b)$$

with $a < b$. For example, $(2, 7)$ is the interval of all numbers between 2 and 7, *not including* 2 and 7.

If a function $f(x)$ is defined only on a closed interval $[a, b]$, it's not obvious what we mean for f to be continuous—mainly because we can only define a one-sided limit (and not a limit) at a and b. But we take what we can get:

Definition 31.9.1. If a function $f(x)$ is defined only on a closed interval $[a, b]$, we say that f is *continuous at a* if

1. The righthand limit $\lim_{x \to a^+} f(x)$ exists, and
2. $\lim_{x \to a^+} f(x) = f(a)$.

Likewise, we say that f is *continuous at b* if

1. The lefthand limit $\lim_{x \to b^-} f(x)$ exists, and
2. $\lim_{x \to b^-} f(x) = f(b)$.

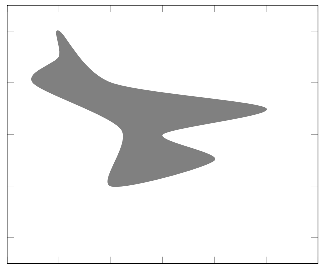
We say that f is continuous if it is continuous at every point of $[a, b]$. [9]

Theorem 31.9.2. The intermediate value theorem holds for continuous functions defined on a closed interval.

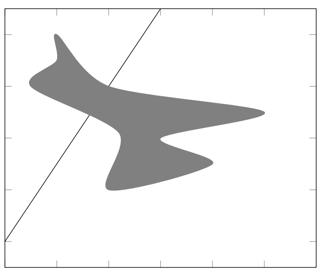
[9] Note that for any element c inside of (a, b)—that is, for any c with $a < c < b$—we know what it means for $f(x)$ to be continuous at c, because we know how to define the limit of f at c.

31.10 A fun exercise: Wonky pizza

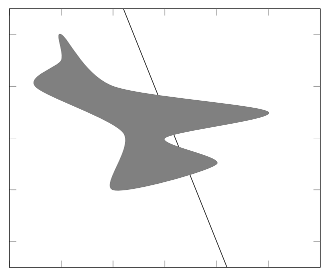
Here is a picture of a wonky-shaped pizza. (And yes, it's gray; not the most tasty-looking thing, is it?)



Your boss wants you to cut this pizza in half, using *one*, linear cut. For example,



and



are two cuts you're allowed to make. Notice that the resulting pizza can have more than just two pieces (as seen on the righthand cut). All that your boss wants is that all the pizza on one side of the cut, has the same area as all the pizza on the other side of the cut.

Exercise 31.10.1. Using the Intermediate Value Theorem, convince yourself that for *any* slope m you choose, you can make a cut of slope m such that you divide the pizza into equal halves (just as your boss requires).

Does the theorem tell you *where* to cut the pizza?

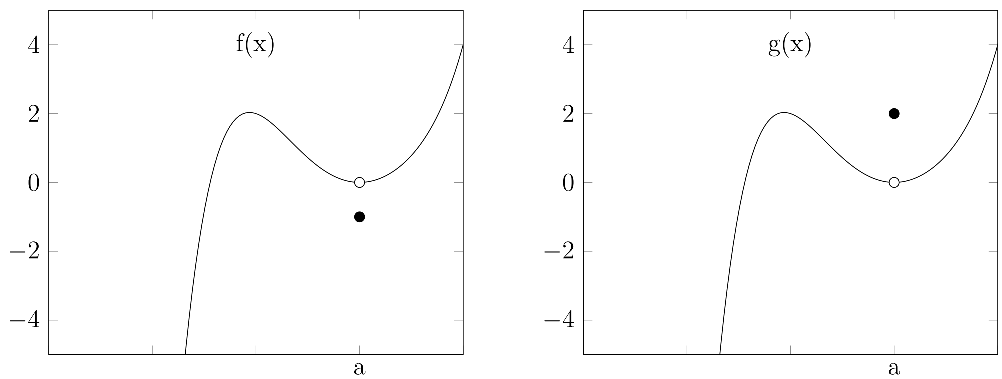
31.11 Puncture law

Let $f(x)$ and $g(x)$ be two functions. Suppose that the two function are equal away from a. Then $f(x)$ has a limit at a if $g(x)$ does, and likewise, $g(x)$ has a limit at a if $f(x)$ does. Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Warning 31.11.1. Many calculus textbooks do *not* talk about a "puncture law." In my opinion, this is a bit ludicrous, because about half of the algebraic "tricks" we have to compute limits are dependent on it. I must admit that I made up the term "puncture law," so you may find your peers outside of your class being confused if you use this law.

Example 31.11.2 (A graphical example)**.** On the left is a graph of $f(x)$, and on the right is a graph of $g(x)$.



Note that the value of $f(x)$ and $g(x)$ are different at a (the black dots are at different heights).[10] But $f(x)$ and $g(x)$ are otherwise identical, so they have the same limit at a. This "obvious" fact is called the puncture law.

Example 31.11.3 (Algebraic example)**.** Let

$$f(x) = \frac{x^2}{x} \qquad \text{and} \qquad g(x) = x.$$

Note that $f(x)$ is not defined at $x = 0$, but is equal to $g(x)$ for all other values of x. Thus, the puncture law tells us that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x). \tag{31.11.1}$$

Of course, you know what the righthand side is (by plugging in what $g(x)$ is):

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x = 0. \tag{31.11.2}$$

[10]Let me remind you—as I mentioned in class—that the white dot means that the function does *not* take the value of the white dot there. The black dot indicates the value of the function. Often, we write a white dot where it looks like a function wants to take a value, but does not.

So, putting (31.11.1) and (31.11.2) together, we see that

$$\lim_{x \to 0} f(x) = 0.$$

In other words (by plugging in the definition of $f(x)$) we find:

$$\lim_{x \to 0} \frac{x^2}{x} = 0.$$

Note that this is an example where the quotient law wouldn't help you, because the limit of the denominator equals zero!

Example 31.11.4 (Rational functions)**.** Let's find the limit

$$\lim_{x \to 2} \frac{(x+1)(x-2)}{x-2}.$$

Note that the function we have is *undefined* when $x = 2$ (because we can't divide by $x - 2$ when $x = 2$). But, we know the following:

$$\frac{(x+1)(x-2)}{x-2} = x + 1 \qquad \textit{so long as } x \neq 1.$$

In other words, the two functions

$$\frac{(x+1)(x-2)}{x-2} \qquad \text{and} \qquad x+1$$

are *equal* away from $x = 1$. Thus, the puncture law tells us

$$\lim_{x \to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x \to 2}(x+1).$$

Now, let's just compute the righthand side:

$$\begin{aligned} \lim_{x \to 2}(x+1) &= \lim_{x \to 2} x + \lim_{x \to 2} 1 \qquad (31.11.3) \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

(We used the addition law in line (31.11.3).) Putting everything together, we conclude:

$$\frac{(x+1)(x-2)}{x-2} = 3.$$

We're done, but let me streamline everything to show you what you might be able to write on a test:

$$\begin{aligned} \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law} \\ &= \lim_{x\to 2} x + \lim_{x\to 2} 1 && \text{by the addition law} \\ &= 2+1 \\ &= 3. \end{aligned}$$

Another solution you might write on a test is:

$$\begin{aligned} \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} &= \lim_{x\to 2}(x+1) && \text{by the puncture law} \\ &= 2+1 && \text{because polynomial functions are continuous} \\ &= 3. \end{aligned}$$

Example 31.11.5 (Another rational function)**.** Let's do another rational function example. Let's compute [11]

$$\lim_{x\to 3} \frac{x^2-2x-3}{x^2-9}.$$

This looks very complicated; to use the puncture law, we'd like to find some other function that is equal to $\frac{x^2-2x-3}{x^2-9}$ away from 3. The trick I want you to learn here is that you can *cancel* $(x-3)$ in the top and bottom. This may seem very confusing, because $(x-3)$ doesn't appear anywhere in the function as it's presented. But you'll see that it *does* appear if you factor.

Pro tip. Why do you want to try to cancel $x-3$? It's because we should feel that a term of the form "$x-3$" is what's causing the denominator to equal zero at $x=3$. So it's natural to try and see if, indeed, a factor of $(x-3)$ can pop up in the denominator. More generally, for rational functions, *if you are computing a limit as x approaches a, it is natural to try to find $(x-a)$ as a factor of the top and bottom.*

Warning. If you don't know how to divide or factor polynomials, you should learn by Googling online and practicing—in this class, you are already *expected* to know how to divide polynomials using long division, or to factor polynomials through other tricks.

In fact, we can factor both the top and the bottom:

$$\frac{x^2-2x-3}{x^2-9} = \frac{(x-3)(x+1)}{(x-3)(x+3)}.$$

[11]Note that the quotient law doesn't help here, because the limit of the denominator equals zero.

And we see that we can cancel the $(x-3)$ terms! So, when x does not equal 3, our function $\frac{x^2-2x-3}{x^2-9}$ is equal to

$$\frac{x+1}{x+3}. \tag{31.11.4}$$

By the puncture law, we thus conclude the following:

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9} = \lim_{x\to 3}\frac{x+1}{x+3}.$$

And, as we saw in the preparation for last lecture, *any rational function is continuous where it is defined.* The rational function in (31.11.4) is defined at $x=3$, so—by the definition of continuity—we can compute the limit simply by plugging 3 into x:

$$\lim_{x\to 3}\frac{x+1}{x+3} = \frac{3+1}{3+3} = \frac{4}{6} = \frac{2}{3}.$$

Putting everything together, we conclude

$$\lim_{x\to 3}\frac{x^2-2x-3}{x^2-9} = 2/3.$$

For next class's quiz, I expect you to be able to use the puncture law to compute limits of rational functions. For example, you should be able to compute the following limits:

1. $\lim_{x\to 0}\frac{x^3+3x^2}{x^2}$.
2. $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$.
3. $\lim_{x\to -2}\frac{x^2-4}{x^2+x-2}$.