Lecture 28

More on epsilon-delta

Let's practice some more epsilon-delta problems. In the next section, you will see some sample problems worked out.

Exercise 28.0.1. Let $f(x) = 3x + 7$. Show that whenever $|x - 1| < \frac{2}{3}$, we can conclude that $|f(x) - 10| < 2$.

(Your answer won't be a number. Instead, your answer will be a string of equalities and inequalities that ultimately show that $|f(x) - 10| < 2$. Put another way, you are being graded for your work!)

Exercise 28.0.2. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{5}$, then $|f(x) - 17| < 1$.

Exercise 28.0.3. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{5}$, then $|f(x) - 17| < 3$. (Yes, every number is the same as the previous problem except for the 3.)

Exercise 28.0.4. Let $f(x) = 3x + 7$. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 10| < \frac{1}{4}$.

Exercise 28.0.5. Let $f(x) = 3x+7$. Suppose somebody gives you a positive number ϵ. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 10| < \epsilon$. (Your δ can be expressed in terms of ϵ.)

Exercise 28.0.6. Let $f(x) = x^2 + 3x + 1$. Suppose $|x - 1| < \frac{1}{12}$. Show that $|f(x) - 5| < \frac{1}{2}$.

Exercise 28.0.7. Let $f(x) = x^2 + 3x + 1$. Can you find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 5| < \frac{1}{4}$?

Exercise 28.0.8. Let $f(x) = x^2 + 3x + 1$. Hiro gives you a number $\epsilon > 0$. Can you find me a number δ so that, whenever $|x-1| < \delta$, we can conclude that $|f(x)-5| < \epsilon$? (Your δ can be expressed in terms of ϵ.)

Exercise 28.0.9. Let $f(x) = 2x^3 + 4x^2 + 7$. Show that if $\delta < \sqrt{\frac{\epsilon}{6}}$, then $|x| < \delta$ implies that $|f(x) - 7| < \epsilon$.

28.1 Sample problems

Exercise 28.1.1. Let $f(x) = 5x + 7$. Show that whenever $|x - 1| < \frac{1}{15}$, we can conclude that $|f(x) - 12| < \frac{1}{3}$.

(Your answer won't be a number. Instead, your answer will be a string of equalities and inequalities that ultimately show that $|f(x) - 10| < 2$. Put another way, you are being graded for your work!)

Solution. Let's first simply $f(x) - 10$ as much as we can.

$$|f(x) - 10| = |5x + 7 - 12| \tag{28.1.1}$$
$$= |5x - 5| \tag{28.1.2}$$
$$= 5|x - 1|. \tag{28.1.3}$$

This number could be huge if x is huge; but we are only asked about "whenever $|x - 1| < \frac{1}{15}$, so let's see what we can conclude when this inequality holds. Well, because $|x - 1| < \frac{1}{15}$, we can multiply this inequality by 5 on both sides to get

$$5|x - 1| < 5 \cdot \frac{1}{15} \tag{28.1.4}$$
$$= \frac{1}{3}. \tag{28.1.5}$$

So tracing through the equalities and inequalities we just worked through, we can conclude that

$$|f(x) - 10| < \frac{1}{3}.$$

Which is what the problem wanted us to show!

In a test, just writing out the lines from (28.1.1) to (28.1.5) would get you full credit. To be safe, you may wan to indicate/write that you used the condition that $|x - 1| < \frac{1}{15}$ in line (28.1.4). □

Exercise 28.1.2. Let $f(x) = 5x + 7$. Show that if $|x - 2| < \frac{1}{4}$, then $|f(x) - 17| < \frac{5}{4}$.

Solution.

$$\begin{aligned} |f(x) - 17| &= |5x + 7 - 17| && (28.1.6) \\ &= |5x - 10| && (28.1.7) \\ &= 5|x - 2| && (28.1.8) \\ &< 5 \cdot \frac{1}{4} && (28.1.9) \\ &= \frac{5}{4}. && (28.1.10) \end{aligned}$$

We used that $|x - 2| < \frac{1}{4}$ in Line (28.1.9). □

Exercise 28.1.3. Let $f(x) = 3x + 3$. Find me a number δ so that, whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 6| < \frac{1}{4}$.

Solution.

$$\begin{aligned} |f(x) - 6| &= |3x + 3 - 6| && (28.1.11) \\ &= |3x - 3| && (28.1.12) \\ &= 3|x - 1|. && (28.1.13) \end{aligned}$$

So we want $3|x - 1|$ to be less than $\frac{1}{4}$. Well, if we want the inequality

$$3|x - 1| < \frac{1}{4}$$

to be true, it's equivalent to wanting the inequality

$$|x - 1| < \frac{1}{12}$$

to be true. So, so long as δ is any number equal to or less than $\frac{1}{12}$, the assumption that $|x - 1| < \delta$ means

$$\begin{aligned} & |x - 1| < \delta && (28.1.14) \\ \implies & 3|x - 1| < 3\delta && (28.1.15) \\ & \leq 3 \cdot \frac{1}{12} && (28.1.16) \\ & = \frac{1}{4}. && (28.1.17) \end{aligned}$$

So I can give you any δ that's less than or equal to $\frac{1}{12}$. For example, $\delta = \frac{1}{12}$, or $\delta = \frac{1}{13}$. For such a δ, the above work shows that whenever $|x - 1| < \delta$, we can conclude that $|f(x) - 6| < \frac{1}{4}$. □

Exercise 28.1.4. Let $f(x) = 3x+6$. Suppose somebody gives you a positive number ϵ. Find me a number δ so that, whenever $|x-2| < \delta$, we can conclude that $|f(x) - 12| < \epsilon$. (Your δ can be expressed in terms of ϵ.)

Solution.

$$|f(x) - 12| = |3x + 6 - 12| \tag{28.1.18}$$
$$= 3|x-6|. \tag{28.1.19}$$

So

$$|f(x) - 12| < \epsilon \tag{28.1.20}$$
$$\iff 3|x-6| < \epsilon \tag{28.1.21}$$
$$\iff |x-6| < \frac{\epsilon}{3}. \tag{28.1.22}$$

In other words, so long as $|x-6| < \frac{\epsilon}{3}$, we are guaranteed that $|f(x) - 12| < \epsilon$. So we can choose δ to equal $\frac{\epsilon}{3}$, or any positive number less than $\frac{\epsilon}{3}$. □

Exercise 28.1.5. Let $f(x) = x^2 + 5x + 1$. Suppose $|x-1| < \frac{1}{12}$. Show that $|f(x) - 7| < \frac{1}{2}$.

Solution.

$$|f(x) - 7| = |x^2 + 5x + 1 - 7| \tag{28.1.23}$$
$$= |x^2 + 5x - 6|. \tag{28.1.24}$$

At this stage, we need to remember a fact I mentioned in class: No matter what, so long as the limit is correct, this polynomial can be factored by $(x-a)$. In our case, $a = 1$ (because we are bounding $|x-1|$ in the problem) and sure enough, we can factor so that

$$|x^2 + 5x - 6| = |(x-1)(x-5)|. \tag{28.1.25}$$

Finally, we want things that look like $|x-1|$ to pop up as much as possible in our expressions. This is because $|x-1| < \frac{1}{12}$ is the only fact we are allowed to use about the number x. So for example, $x-5$ can be re-written to be $(x-1)-4$. So let's do that.

$$|(x-1)(x-5)| = |(x-1)((x-1)-4)|. \tag{28.1.26}$$

Next, remember that $|AB| = |A| \cdot |B|$. So

$$|(x-1)((x-1)-4)| = |x-1| \cdot |(x-1)-4|. \tag{28.1.27}$$

Finally, we use the triangle inequality, which tells us that $|C + D| \leq |C| + |D|$. So

$$|(x-1)-4| \leq |x-1| + |4|. \tag{28.1.28}$$

Multiplying both dies of this inequality by $|x-1|$, we see that

$$|x-1| \cdot |(x-1)-4| \leq |x-1| \cdot (|x-1|+4). \tag{28.1.29}$$

We have come a long way to find that (by tracing through all the equations above, along with the one inequality):

$$|f(x)-7| \leq |x-1| \cdot (|x-1|+4). \tag{28.1.30}$$

Because the doodle on the right, $(|x-1|+4)$, is always bigger than or equal to $|f(x)-7|$, if we can guarantee that this doodle is less than ϵ, then we know that $|f(x)-7|$ is also less than ϵ.

Well, we are told that $|x-1|$ is less than $\frac{1}{12}$. So let's see what happens to the doodle:

$$|x-1| \cdot (|x-1|+4) < \frac{1}{12} \cdot \left(\frac{1}{12}+4\right). \tag{28.1.31}$$

Whatever is in the parentheses is certainly smaller than $1+5$, so we can write that

$$\frac{1}{12} \cdot \left(\frac{1}{12}+4\right) < \frac{1}{12} \cdot 5 \tag{28.1.32}$$

$$= \frac{5}{12}. \tag{28.1.33}$$

On the other hand,

$$\frac{5}{12} < \frac{6}{12} \tag{28.1.34}$$

$$= \frac{1}{12}. \tag{28.1.35}$$

Combining all the above, line by line, we conclude that $|f(x)-7| < \frac{1}{12}$, as desired.

(Your solution is the entirety of the work above!) □

Exercise 28.1.6. Let $f(x) = x^2 + 4x + 1$. Hiro gives you a number $\epsilon > 0$. Can you find me a number δ so that, whenever $|x - (-1)| < \delta$, we can conclude that $|f(x) - (-2)| < \epsilon$? (Your δ can be expressed in terms of ϵ.)

Solution.

$$\begin{aligned}
|f(x) - (-2)| &= |x^2 + 4x + 1 + 2| && (28.1.36)\\
&= |x^2 + 4x + 3| && (28.1.37)\\
&= |(x+1)(x+3)| && (28.1.38)\\
&= |x+1||(x+1)+2| && (28.1.39)\\
&\leq |x+1|\left(|x+1|+2\right). && (28.1.40)
\end{aligned}$$

Let's suppose that $|x+1|$ is less than some number C, so $|x+1| < C$. Then we can conclude that

$$|x+1|\left(|x+1|+2\right). < |x+1|\left(C+2\right). \qquad (28.1.41)$$

If further $|x+1|$ is less than $\frac{\epsilon}{C+2}$, we conclude that

$$\begin{aligned}
|x+1|\left(C+2\right) &< \frac{\epsilon}{C+2}\cdot(C+2) && (28.1.42)\\
&= \epsilon. && (28.1.43)
\end{aligned}$$

So, for any positive number C, choose δ to be any positive number less than C and less than $\frac{\epsilon}{C+2}$. Then the work above guarantees that $|x+1| < \delta$ guarantees that $|f(x) - (-2)| < \epsilon$.

So for example, we could choose δ to be any number less than 1 and less than $\frac{\epsilon}{3}$. □

Exercise 28.1.7. Let $f(x) = 2x^3 + 4x^2 + 3$. Show that if $\delta < \sqrt{\frac{\epsilon}{6}}$ and if $\delta < 1$, then $|x| < \delta$ implies that $|f(x) - 3| < \epsilon$.

Solution.

$$\begin{aligned}
|f(x) - 3| &= |2x^3 + 4x^2 + 3 - 3| && (28.1.44)\\
&= |2x^3 + 4x^2| && (28.1.45)\\
&= 2|x^3 + 2x^2| && (28.1.46)\\
&= 2|x^2||x+2|. && (28.1.47)
\end{aligned}$$

We are asked to show that the condition $|x| < \delta$ implies something. Well, if $|x| < \delta$, then—given what we know about δ—we conclude that $|x| < \sqrt{\frac{\epsilon}{6}}$ and $|x| < 1$. So

$$\begin{aligned}
2|x^2||x+2| &< 2\left(\sqrt{\frac{\epsilon}{6}}\right)^2(1+2) && (28.1.48)\\
&= 2\left(\frac{\epsilon}{6}\right)(3) && (28.1.49)\\
&= \epsilon. && (28.1.50)
\end{aligned}$$

The string of equalities and inequalities above shows that $|f(x) - 3| < \epsilon$, as desired. $\square$