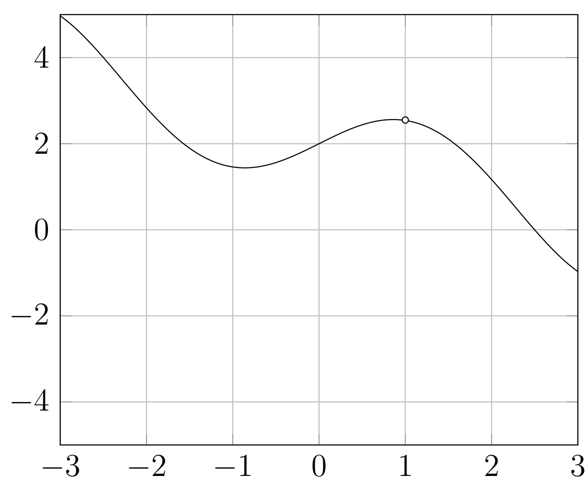
Lecture 27

Limit laws

Today, we're going to work on *building intuition and tools*, rather than attacking the formal definition of a limit.

27.0.1 You can ask for limits at various points

Consider the following function $f(x)$:



It is not defined at $x = 1$, but it clearly has a limit there. So, though we have only been talking at limits as x approaches zero, we see the following:

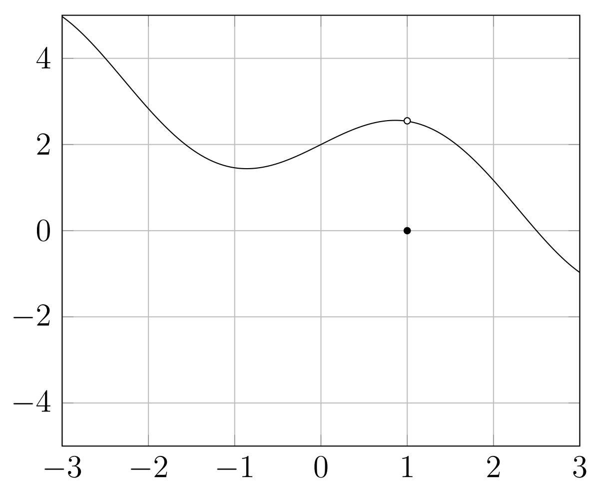
We can ask about the limit of $f(x)$ as x approaches some number other than zero, too.

Whenever the limit exists as x approaches a, we will write this limit as

$$\lim_{x \to a} f(x).$$

27.0.2 Limits where the function is defined

Consider the following function $f(x)$:

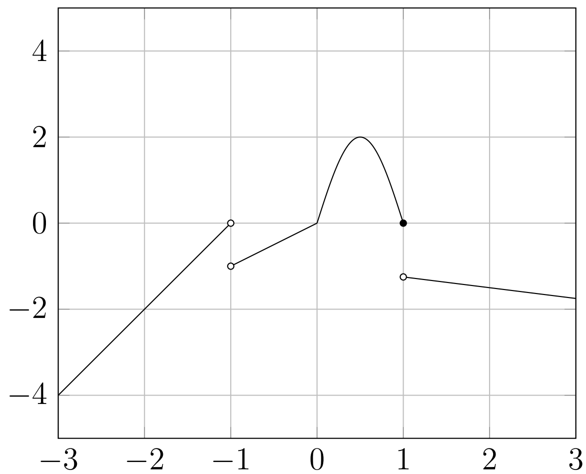


This time, $f(x)$ *is* defined at $x = 1$. (We see $f(x) = 0$.) But we also see that the limit of $f(x)$ as x approaches seems to *want* to be something like 2.5—not the value of f. Thus, we see the following:

It is possible that $\lim_{x \to a} f(x)$ exists, and is not equal to $f(a)$.

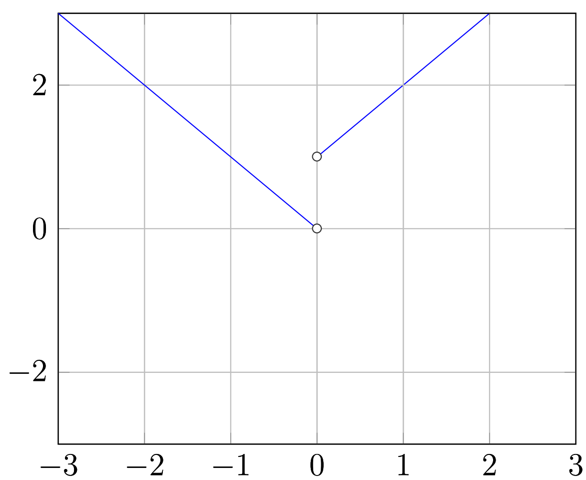
27.0.3 Limits might not exist

And as we have already seen, some functions may not have limits at certain points. Below is an example of a function that does not have limits at the points -1 and 1:



27.1 One-sided limits

Consider a function f whose graph looks as follows:



f is not defined at $x = 0$. It also does not have a limit at $x = 0$. However, it's clear that f wants to approach a certain height if we approach $x = 0$ from the left, and from the right. (The two heights don't agree.)

We will denote by

$$\lim_{x \to x_0^+} f(x)$$

the height that f wants to approach as x approaches x_0 *from the right*. We let

$$\lim_{x \to x_0^-} f(x)$$

denote the height/value that f approaches as x approaches x_0 from the left. You may make use of the following proposition from hereon:

Proposition 27.1.1. $\lim_{x \to x} f(x) = L$ if and only if both $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ equal L.

More precisely, f has a limit at x_0 if and only if both one-sided limits exist at x_0, and if these limits agree.

27.2 Limit laws: The straightforward ones

Today I'm going to tell you that you can rely on certain laws for computing limits.

Remark 27.2.1. These laws are dissatisfying, because you should demand more: *Why* are these laws valid? We will why later, when we apply the ϵ-δ definition to *prove* these laws.

Limits of constants. If $f(x)$ is a constant function[1] with value C, then

$$\lim_{x \to a} f(x) = C$$

regardless of a.

Limits of x. For the function $f(x) = x$, we have that

$$\lim_{x \to a} f(x) = a.$$

(I encourage you to graph the function $f(x) = x$; then this law will seem "obvious" to you.)

Remark 27.2.2. The first two laws are hopefully not too bewildering; the notation is confusing, but these are meant to be among the simplest examples. I state these just to get our feet wet; it's the next few laws that will really get us going.

[1] This means $f(x) = C$ for some number C. Put another way, the graph of $f(x)$ is just a flat, horizontal line.

Limits scale. *If a limit already exists*, then the limit of the scaled function is the scaled limit of the function. More precisely: *If* $\lim_{x\to a} f(x)$ *already exists*, then for any number m, we have the following:

$$\lim_{x\to a} (m \cdot f(x)) = m \cdot \left(\lim_{x\to a} f(x)\right)$$

Limits add. *If the limits already exist,* then the limit of the sum exists; moreover, the sum of the limits is the limit of the sum.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) + g(x))$ exists, and

$$\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$$

Limits multiply. *If limits already exist,* then the limit of the product exists; moreover, the product of the limits is the limit of the product.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x) \cdot g(x))$ exists, and

$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$

Limits divide. *If limits already exist,* then the limit of the quotient exists; moreover, the quotient of the limits is the limit of the quotient (provided the denominator is not zero).

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} (f(x)/g(x))$ exists, and

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

so long as $\lim_{x\to a} g(x) \neq 0$.

Remark 27.2.3. The above limit laws have three parts: (i) The *given knowledge* that certain limits already exist, (ii) The *guarantee* that another limit exists, and (iii) The *formula* of how to compute that other limit.

I wrote all the formulas in such a way that the righthand side of the formula consists of the limits given to exist; the lefthand side is the limit that we are then guaranteed to exist.

Remark 27.2.4. It's important to note that, for every law, the limits are taken at the same point. That is, every limit in sight is taken as x approaches a single number a. So for example, even if I know that $\lim_{x\to a} f(x)$ exists, and that $\lim_{x\to b} g(x)$ exists, I don't know anything about the limits of $f(x) + g(x)$ unless $a = b$. (In which case, I know that a limit exists as $x \to a$.)

27.2.1 Practice with the straightforward limit laws

All of the exercises below could have been solved by "looking at the graphs." But I want you to instead solve them by using the limit laws.

Exercise 27.2.5. Using some of the facts above, convince yourself that if $g(x) = mx$, then[2]

$$\lim_{x\to a} g(x) = g(a).$$

(Hint: Use the function $f(x) = x$ and the scaling law.)

Exercise 27.2.6. Using some of the facts above, convince yourself that if $h(x) = x^2$, then

$$\lim_{x\to a} h(x) = h(a).$$

(Hint: Use the functions $f(x) = x$ and $g(x) = x$, along with the product law.)

Exercise 27.2.7. Using some of the facts above, convince yourself that if $h(x) = x^2 + 3$, then

$$\lim_{x\to a} h(x) = h(a).$$

Exercise 27.2.8. Using some of the facts above, show that **limits subtract**.

More precisely, if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a} (f(x) - g(x))$. Moreover,

$$\lim_{x\to a} (f(x) - g(x)) = \left(\lim_{x\to a} f(x)\right) - \left(\lim_{x\to a} g(x)\right)$$

(Hint: Use the fact that limits scale, taking your scaling constant to be $m = -1$, and use the fact that limits add.)

Exercise 27.2.9. Use the limit laws to compute

$$\lim_{x\to 1} \left(\frac{x^2 + 3}{x}\right).$$

What goes wrong when you try to compute the limit as $x \to 0$?

[2] The graph of $g(x)$ is a line of slope m with zero as y-intercept. So even before you knew these limits laws, you should have been able to tell me what the limit as $x \to a$ is!

27.3 Continuity

You saw examples of functions $f(x)$ that satisfied what seems like a nice property:

$$\lim_{x \to a} f(x) = f(a).$$

That is, the limit of the function at a is actually the *value* of the function at a. (Implicit here is that the function also *has* a limit at a, which is already a nice property of the function.)

This isn't always the case for all functions. For example, we saw in Section 27.0.2 an example of a function that has a limit as $x \to 1$, but whose limit there doesn't equal $f(1)$.

So let's have an adjective to describe this "nice" property: continuous.

Definition 27.3.1. A function f is called *continuous at a* if

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

Remark 27.3.2. This is a definition for the phrase "continuous at a."

Example 27.3.3. Let $f(x) = 1/x$. Then f is not continuous at zero because it is not defined at $x = 0$.

Example 27.3.4. Let $f(x) = 1/x$. Then f *is* continuous at 2. To see this, we just need to check all three conditions in Definition 27.3.1.

1. Clearly, $f(2)$ is defined. Then,
2. the fact that limits divide (one of the limit laws!) tells us that $\lim_{x \to 2} f(x)$ exists, and
3. this limit is computed as

   $$\lim_{x \to 2} f(x) = \frac{\lim_{x \to 2} 1}{\lim_{x \to 2} x} = \frac{1}{2}.$$

   On the other hand $f(2) = 1/2$ by definition of $f(x)$, so we see that

   $$\lim_{x \to 2} f(x) = f(2).$$

So we have checked that all three conditions of continuity in Definition 27.3.1 are satisfied. This means $f(x) = 1/x$ is continuous at $x = 2$.

In preparation for next lecture, you should be able to answer the following questions:

(a) What are the *three conditions* you need to check to see whether a function $f(x)$ is continuous at a?

(b) Why is $f(x) = 3/x$ not continuous at zero?

(c) Why is $f(x) = 8/x$ continuous at 5?