Lecture 26

Limits and ϵ-δ

26.1 Remembering derivatives

When estimating slopes, we were led to a natural question: Does the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

tend to, or approach, some number as h approaches zero?

Then we defined the following:

Definition 26.1.1 (The derivative)**.** If the difference quotient tends to some number $f'(x)$ as h approaches zero, we call $f'(x)$ the *derivative* of f at x.

Example 26.1.2. The difference quotient for $f(x) = x^2+2$, and $x = 1$, is as follows:

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+2-(x^2+2)}{h} \tag{26.1.1}$$

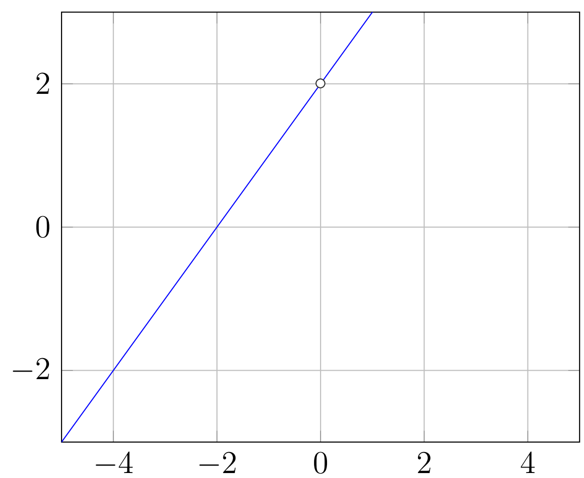
$$= \frac{x^2+2hx+h^2+2-x^2-2)}{h} \tag{26.1.2}$$

$$= \frac{2hx+h^2}{h} \tag{26.1.3}$$

$$= \begin{cases} 2x+h & \text{when } h \neq 0 \\ \text{undefined} & \text{when } h = 0 \end{cases} \tag{26.1.4}$$

$$= \begin{cases} 2+h & \text{when } h \neq 0 \\ \text{undefined} & \text{when } h = 0 \end{cases} \tag{26.1.5}$$

So the difference quotient is a function depending on h, and it is undefined when $h = 0$ (because we can't divide by zero). We can draw the graph of this function (which depends on h) as follows:



The horizontal axis is labeled by h. The circle in the middle of the graph represents a place where the graph *does not* pass through. In other words, that circle at the point $(0, 2)$ is *not* part of the graph (because the difference quotient is not defined at $h = 0$).

However, as h tends to zero, there is a clear value that this function "wants" to take. It is 2. Thus, we observe:

The derivative of $f(x) = x^2 + 2$ at $x = 1$ should be 2.

(End of example.)

But this is no way to live life. We shouldn't have to draw the graph of a difference quotient, and fill in a hole at $h = 0$, each time we want to find the slope of f at some point x.

So here's what we're going to do: Suppose we are given a function

$$q(h)$$

that is defined everywhere except at $h = 0$. We'd like to explore basic examples and tricks to determine whether $q(h)$ approaches some value as h goes to zero. If such a value exists, it will be called *the limit of $q(h)$ as h goes to zero.* This limit will be written

$$\lim_{h \to 0} q(h).$$

So let's get some practice.

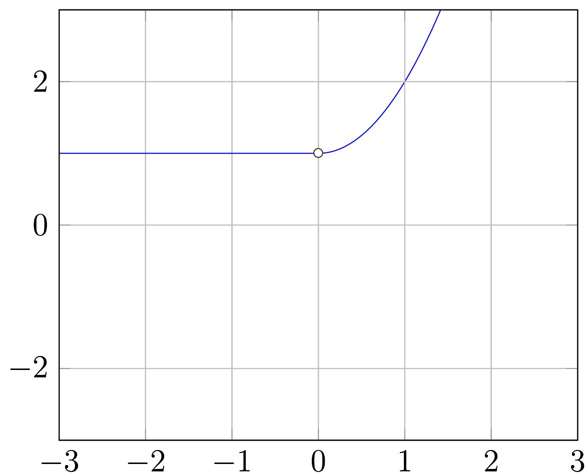
26.2 Limits, visually

Below are some graphs of functions $q(h)$. Each function q is defined everywhere except at $h = 0$. For each, determine whether the limit

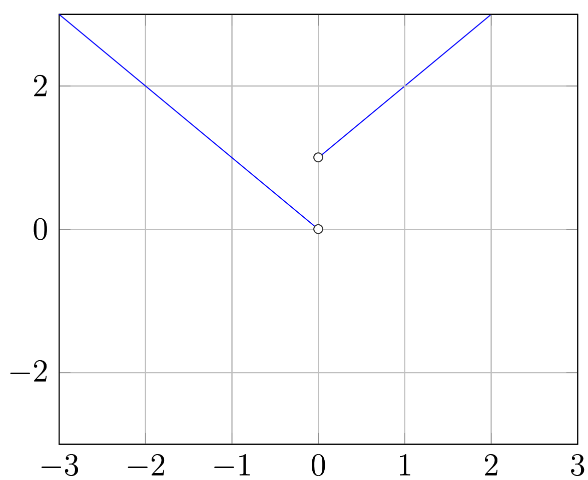
$$\lim_{h \to 0} q(h)$$

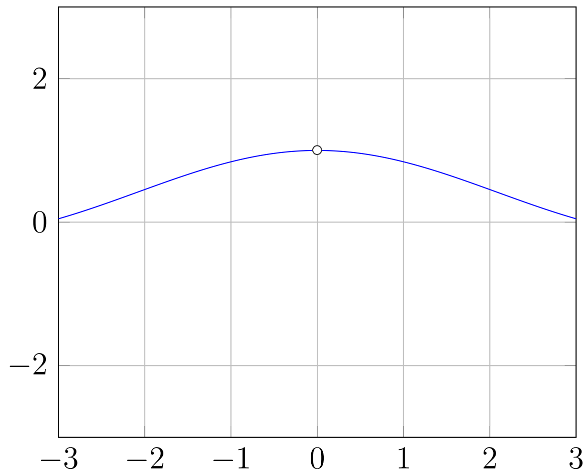
exists; and if so, say what the limit is.

(a)



(b)





(c)

In the above examples, (a) and (c) have functions whose graphs clearly "approach" a particular point on the xy-plane as we move along the blue curve toward $h = 0$. In both cases, that point has height 1, so we would expect that the limit $\lim_{h\to 0} q(h) = 1$.

In (c), if we approach $h = 0$ from the left, it looks like $q(h)$ wants to attain the value 0. On the other hand, approaching $h = 0$ from the right, $q(h)$ approaches the value 1. There is *not* a single value that $q(h)$ approaches, so we say that the limit does not exist.

26.3 Limits for functions that aren't presented visually

Below are some functions $q(h)$. Each function q is defined everywhere except at $h = 0$. For each, determine whether the limit

$$\lim_{h\to 0} q(h)$$

exists; and if so, say what the limit is.

1. $q(h) = \begin{cases} h^2 & h \neq 0 \end{cases}$

2. $q(h) = \begin{cases} \sin(h) & h > 0 \\ \cos(h) & h < 0 \end{cases}$

3. $q(h) = \begin{cases} 1 & h \text{ is a rational number and } h \neq 0 \\ 0 & h \text{ is an irrational number} \end{cases}$

Remark 26.3.1. Recall that a rational number is a number that can be expressed as a fraction—things like $-2/7$, or 13, or $5/6$. An irrational number is a real number that is not a fraction. For example, $\sqrt{2}$ or π.)

Remark 26.3.2. A function is called *piecewise defined* when it is defined in the following format:

$$q(h) = \begin{cases} \text{blah blah} & \text{some condition on } h \\ \text{blahbitty blah} & \text{some other condition on } h \\ \text{Rob Loblaw} & \text{perhaps another condition on } h \end{cases}$$

We tend to define functions using the above format when it's not easy to define the function in one fell swoop. For example, the function (3) above means that $q(h)$ equals 1 when h is a non-zero rational number, and equals 0 when h is an irrational number.

26.4 Epsilon-delta condition

For the next few days, we will explore something called ϵ-δ proofs (this is read as "epsilon-delta" proofs).

Here is the general principle: Given a function g and a suspected limit for g, you must find a δ (read *delta*) that guarantees you can get within ϵ (*epsilon*) of the suspected limit after applying g.

Example 26.4.1. Let $g(x) = (8x^2 + x)/x$. You suspect that the limit of $g(x)$ as x approaches zero is 1. (You might arise at such a suspicion by simplifying g, or drawing a graph of g.)

Now let $\epsilon = 0.1$. Can you find a positive number δ so that, so long as you choose a $x \neq 0$ with $|x| < \delta$, then $f(x)$ is within ϵ of 1? (Put another way, so long as x is small enough—meaning its absolute value is less than δ—then the value of $g(x)$ is very close to 1—meaning at most distance ϵ from 1.)

Yes, you can.

To see how you can find this δ, let's note the following:

$$|g(x) - 1| = |\frac{8x^2 + x}{x} - \frac{x}{x}| = |\frac{8x^2 + x - x}{x}| = |\frac{8x^2}{x}| = |8x| \qquad \text{(when } x \neq 0\text{)}$$

The very lefthand side of this expression is the distance between $g(x)$ and the suspected limit, 1. The very righthand side is telling you that this distance is always given by $|8x|$ when $x \neq 0$. So for example, if you took x to be 0.2, then the distance between $g(x)$ and your suspected limit would be $|8x| = |8 \times 0.2| = 1.6$.

So if you want $g(x)$ to be within ϵ of 1, you want $|8x|$ to be less than ϵ. That is, you want

$$|8x| < \epsilon.$$

This happens so long as $|x| < \epsilon/8$. So, choose $\delta = \epsilon/8$. Then so long as $|x| < \delta$, you can guarantee that $|g(x) - 1| < \epsilon$.

Note that while I originally asked for a δ so that you are within 0.1 of the suspected limit, you have discovered that regardless of ϵ, you can choose $\delta = \epsilon/8$ to be within ϵ of the suspected limit.

Example 26.4.2. Let $g(x) = 3x/x$. You suspect that the limit of $f(x)$ as x approaches zero is 3.

And let $\epsilon = 12$. Can you find a positive number δ so that, so long as $x \neq 0$ and $|x| < \delta$, then $g(x)$ is within ϵ of 3?

Yes; in fact, any positive number δ will do. This is because—regardless of x—$g(x)$ is always equal to 3 so long as $x \neq 0$. Thus

$$|g(x) - 3| = |\frac{3x}{x} - 3| = 0 \qquad \text{whenever } x \neq 0$$

and 0 is of course smaller than any ϵ. So, regardless of δ, your $g(x)$ will always be within δ of 3.

Example 26.4.3. Let $g(x) = (4x^3 + 9x)/x$. You suspect that the limit of $g(x)$ as x approaches zero is 9. (You might arise at such a suspicion by simplifying g, or drawing a graph of g.)

Now suppose someone gives you some positive number called ϵ. Can you find a positive number δ so that, so long as you choose a value of x so that $x \neq 0$ and $|x| < \delta$, then $g(x)$ is within ϵ of 9? That is, can you find a δ so that

$$|x| < \delta, x \neq 0 \qquad \text{implies} \qquad |g(x) - 9| < \epsilon?$$

Yes, you can.

To see how you can find this δ, let's note the following:

$$|g(x) - 9| = |\frac{(4x^3 + 9x)}{x} - \frac{9x}{x}| = |\frac{4x^3 + 9x - 9x}{x}| = |\frac{4x^3}{x}| = |4x^2| \qquad (\text{when } x \neq 0)$$

The very lefthand side of this expression is the distance between $g(x)$ and the suspected limit, 9. The very righthand side is telling you that this distance is always given by $|4x^2|$ when $x \neq 0$. So for example, if you took x to be 0.1, then the distance between $g(x)$ and your suspected limit would be $|4x^2| = |4 \times 0.01| = 0.04$.

So if you want $g(x)$ to be within ϵ of 9, you want $|4x^2|$ to be less than ϵ. That is, you want

$$|4x^2| < \epsilon.$$

That is, you want

$$|x^2| < \epsilon/4.$$

Because squaring a number preserves $<$—meaning $a^2 < b^2$ if and only if $|a| < |b|$—we conclude that for the above inequality to hold, we want

$$|x| < \sqrt{\epsilon/4}.$$

Thus, set $\delta = \sqrt{\epsilon/4}$. Then, based on the work above, we know that if $|x| < \delta$, then $|g(x) - 9|$ is less than ϵ.

26.5 For next time

For next quiz, you will be tested on whether—given $g(x)$, a suspected limit L, and ϵ—you can find a δ so that

$$\text{If } x \neq 0 \text{ and } |x| < \delta, \text{ then } |g(x) - L| < \epsilon.$$

You will be quizzed on the following g and L. (You should be able to find δ as an expression involving only g, L, ϵ, though often you will not need L at all.)

1. $g(x) = (2x^3 + 9x)/x$, with $L = 9$.

2. $g(x) = (5x^2 + 7x)/x$, with $L = 7$.

3. $g(x) = 3$, with $L = 3$.