Lecture 16

Implicit differentiation

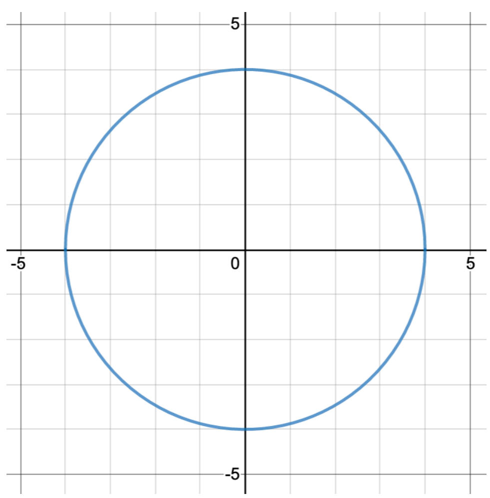
Today, we're going to talk about a computational technique called *implicit differentiation*.

The images in this lecture were generated using Desmos.

16.1 There are shapes that aren't graphs

We've practiced taking derivatives of $f(x)$. Today, we're going to expand our minds. Here are two examples to keep in mind.

Example 16.1.1. Consider the circle of radius 4:



As you know, the circle is *not* the graph of any function. For example, the circle fails the vertical line test.[1]

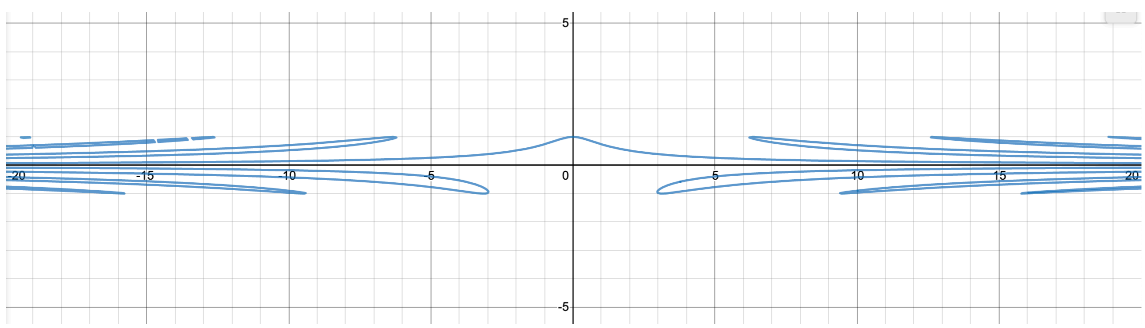
Regardless: If you know *where* you are on the circle, can you know the slope of the tangent line to the circle there?

Example 16.1.2. In science, it happens all the time that we look for solutions to equations like the following:

$$y - \cos(xy) = 0.$$

The key point here is that the appearances of y *cannot be separated* from the functions and variables; so it is either difficult, or impossible, to put the above equations into the form $y =$ (something involving only x). So we'll rarely find that the set of all points satisfying the above equation is a graph of something.

Can you plot all the points (x, y) on the plane so that the above equation is satisfied? What does the shape look like? This turns out to be very hard; in case you're curious, here's a bit of the solution set. It looks even more interesting as you zoom out from what I've drawn here.



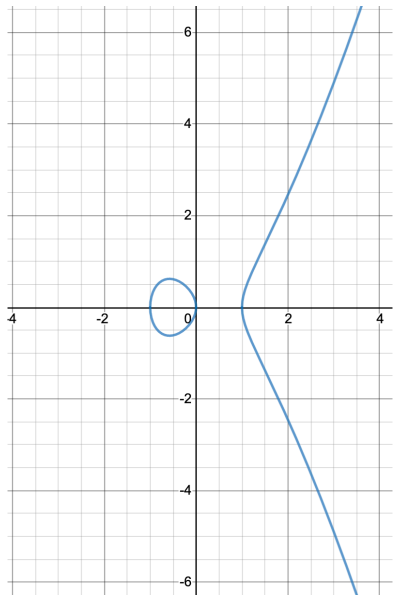
(This solution set is definitely not the graph of some function; it fails the vertical line test.)

Regardless, let's say you can find some point that solves the above equation. Can you at least find the slope (of the tangent line) at that point? Then, even if you can't visualize the above shape, you can still see very interesting information!

Example 16.1.3. Another example is below; it's something called an *elliptic curve*, and in this case, we're plotting all those points (x, y) satisfying

$$y^2 = x^3 - x.$$

[1]I expect you to know about the vertical line test from precalculus.



16.2 The technique

Implicit differentiation *pretends* that y is a function of x, and then takes the derivative. Let me say what I mean.

Assume you have a function f, and that you know the function satisfies the following equation for all x:

$$f(x) - \sin(xf(x)) = 12.$$

Well, this says that there's a function on the lefthand side, and a (constant) function on the righthand side, and they're equal; so their derivatives must be equal! Let's take the derivatives of both sides.

$$f'(x) - \cos(xf(x))(f(x) - xf'(x)) = 0.$$

(On the left, I've used the chain rule.) We can rearrange terms to find:

$$f'(x) = \frac{f(x)\cos(xf(x))}{1 - x}.$$

In other words, we have found the derivative of f *in terms of* x *and* $f(x)$—if we know x and we know $f(x)$, we know the slope of f there.

So suppose instead the you are curious about the shape formed by the equation

$$y - \sin(xy) = 12.$$

We are going to **pretend y is a function of x**, and we will take the derivatives of both sides:

$$y' - \cos(xy)(y - xy') = 0.$$

Then, we solve for y':

$$y' = \frac{y\cos(xy)}{1-x}.$$

Note that this answer is *identical* to the above answer, with $f(x)$ replaced by y. Here is how to interpret this equation: On the lefthand side is the slope of my shape, and on the righthand side is an expression for that slope in terms of x and y. Put another way, *if I know where I am, I know the slope of my shape there.* Here, "where I am" is given by the value of x and y I plug into the righthand side—it's given by the point (x, y) on the plane.

Example 16.2.1. Consider the ellipse given by the equation

$$3\,(x-3)^2 + (y-1)^2 = 2.$$

Find the slope of the tangent line to the ellipse at a point (x, y) on the ellipse.

Here is the solution: We take the derivative of both sides of the above equation, pretending that y is a function of x. Then we get

$$\left(3\,(x-3)^2 + (y-1)^2\right)' = (2)'.$$

The lefthand side becomes

$$\left(3\,(x-3)^2 + (y-1)^2\right)' = (3\,(x-3)^2)' + ((y-1)^2)' = 6(x-3) + 2(y-1)y'.$$

Thus

$$6(x-3) + 2(y-1)y' = 0.$$

Now we rearrange the equation so that y' is alone:

$$y' = \frac{-3(x-3)}{y-1}.$$

This gives the answer.

For example, you can check that the point $(3, 1+\sqrt{2})$ is on this ellipse. Then the slope of the tangent line there is given by

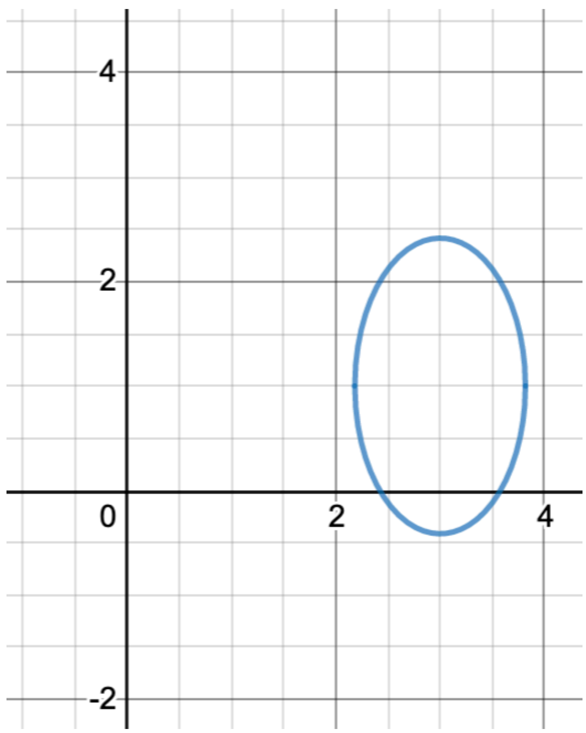
$$\frac{-3(3-3)}{1+\sqrt{2}-1} = 0.$$

You can also check that the point $(3+\sqrt{1/3}, 2)$ is on this ellipse. The slope of the tangent line there is given by

$$\frac{-3(3+\sqrt{1/3}-3)}{2-1} = -3\sqrt{1/3}.$$

Note also that y' approaches infinity as y approaches 1. Indeed, these are points at which the tangent line becomes vertical.

Here is a picture of the ellipse in case you want to study our results further:



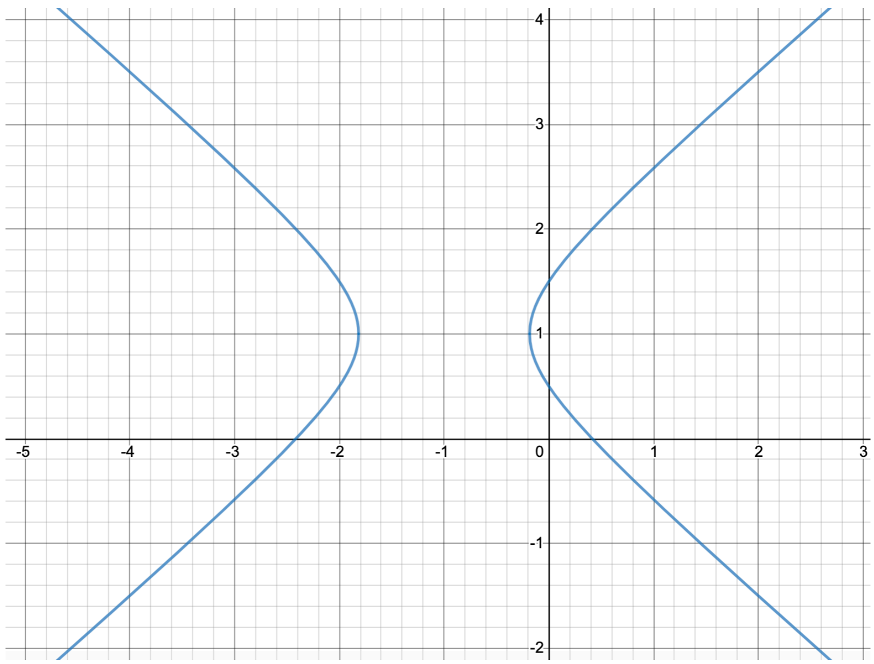
Exercise 16.2.2. Here is an equation for a hyperbola:

$$3\,(x+1)^2 - 4\,(y-1)^2 \;=\; 2.$$

(i) Using implicit differentiation, find a formula for the slope of the hyperbola in terms of the x and y coordinates.

(ii) How does this slope behave as x approaches ∞? (Is there a single behavior?) Be warned: This is a fun problem and will take a little trickery!

(iii) Below is an image of the hyperbola. What does your answer to part (ii) have to do with this picture?



16.3 For next time

You should be able to do the following problem (and problems similar to it):

Consider the collection of all points (x, y) on the xy-plane satisfying the equation

$$(x-3)^2 + (y-3)^2 = 9.$$

(a) Write down a formula for the slope of the tangent line to this shape at a point (x, y) on this shape.

(b) Find the slope of the tangent line to this shape at the point $(0, 6)$.

(c) Find the slope of the tangent line to this shape at the point $(1, 3 + \sqrt{5})$.