Lecture 14

Related Rates

Sometimes in life, a quantity f depends on another quantity g, and in turn, g depends on x. In such instances, we'd like to understand how f depends on x (skipping the "middle man" g).

Example 14.0.1. A submarine is descending into water. The depth d of the submarine depends on how long it has been descending, so we can write the depth as a function of time. We will write this as $d(t)$.

On the other hand, the *pressure* that the submarine experiences underwater depends on the depth that the submarine is at. Writing P for pressure, we can express pressure as a function $P(d)$, depending on depth.

So, if we want to re-write pressure as a function of time, we study the composite

$$P \circ d.$$

This takes an input time t, and outputs the pressure that the submarine experiences at time t.

Example 14.0.2. The opacity (opacity is the opposite of transparency) of the air above California is a function of the concentration of certain compounds in the atmosphere. Let's write this as a function $O(C)$, where O stands for opacity, and C stands for concentration.

Now-a-days, the concentration of certain compounds is a function of how far you are from the nearest forest fire. So we may write the concentration function, as a function of distance, as $C(d)$.

Then the composition $O(C(d))$, or $(O \circ d)(d)$, represents the opacity of the air as a function of how far you are from the nearest forest fire.

Example 14.0.3. The average global temperature of the earth depends on the amount of carbon that is present in the atmosphere. So we can write the temperature T as a function of the total amount C of carbon in the atmosphere.

On the other hand, we can make different models for the amount of carbon in the atmosphere as a function of time. (So for example, we this model would look different if the US had stayed in the Paris Agreement. It would look different if many nations opted out of the Paris Agreement.) Using these models, we can understand the average global temperature T as a function of time t. This is the function $T(C(t))$, or $(T \circ C)(t)$.

Thanks to the chain rule, if we know how the "inside function" changes with respect to the input variable (e.g., if we know the derivative of the inside function), and if we know the derivative of the outside function, we can compute the rate of change (the derivative) of the composite function!

For whatever reason, problems that involve computing rates of change of composite functions are called *related rates* problems.

14.1 Examples of related rates problems (using that area depends on radius)

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Exercise 14.1.1. The area of a crop circle is expanding at a rate of 3 meters squared per minute (i.e., $3\text{m}^2/\text{min}$). If you know what the radius of the crop circle is at a certain time, can you tell me how quickly the radius of this crop circle increasing at that time?

Exercise 14.1.2. The area of a different crop circle at time t is given by

$$A(t) = e^{3t},$$

where t is in minutes and the area is in meters squared. At time t, how quickly is the radius of this crop circle increasing?

In tackling each of these problems, we have to think about how the area depends on the radius. Of course, for circles, we know that area is equal to π times the radius squared. That is,

$$A = \pi r^2.$$

Now, the area is changing with time, so the radius is changing with time, too. We can write

$$A(t) = \pi r(t)^2.$$

(Both area and radius are now expressed as functions of time.)

So let's take the derivative of both sides, with respect to t:

$$A'(t) = \pi 2r(t) \cdot r'(t).$$

(We are using the chain rule on the righthand side!) Dividing both sides by $2\pi r(t)$, we find:

$$r'(t) = \frac{A'(t)}{2\pi r(t)}.$$

So, for the first exercise, when the circle has radius R, we know that the radius is changing as

$$r'(t) = \frac{A'(t)}{2\pi r(t)} = \frac{3}{2\pi R}.$$

For the second exercise, we see that

$$A'(t) = (e^{3t})' = 3e^{3t}.$$

Moreover, we can find $r(t)$ in terms of $A(t)$:

$$r(t) = \sqrt{A(t)/\pi} = \sqrt{e^{3t}/\pi} = \frac{e^{3t/2}}{\sqrt{\pi}}.$$

So

$$\begin{aligned} r'(t) &= \frac{A'(t)}{2\pi r(t)} && (14.1.1) \\ &= \frac{3e^{3t}}{\frac{e^{3t/2}}{\sqrt{\pi}}} && (14.1.2) \\ &= \frac{3e^{3t-(3t)/2}}{\frac{1}{\sqrt{\pi}}} && (14.1.3) \\ &= 3\sqrt{\pi}e^{3t/2}. && (14.1.4) \end{aligned}$$

We can go the other direction, too:

Exercise 14.1.3. A culture of bacteria is growing on a petri dish. At any time t, the bacteria are taking up a circular region on the petri dish, and the radius of this region is modeled by the following function:

$$r(t) = 8t$$

where t is in seconds and r is in micrometers.

At $t = 3$ seconds, how quickly is the area of the circular region changing, in units of micrometers-squared-per-second?

Exercise 14.1.4. A teardrop falls onto a lake, and the resulting ripple grows in radius. The radius can be modeled as a function of time as follows:

$$r(t) = 5e^{-3t}$$

where r is in centimeters and t is in seconds.

In terms of centimeters-squared-per-second, how quickly is the area enclosed by the ripple growing at $t = 2$ seconds?

Exercise 14.1.5. A cube is forming in outer space. The length of one edge of the cube is changing a function of time as follows:

$$E(t) = e^t \sin(t)^2.$$

You will need to use the fact that the volume of a cube of edge length E is given by E^3.

14.2 Related rates problems involving multiple dependencies

The following are a bit more challenging.

Exercise 14.2.1. Both the radius and length of a cylinder-shaped popsicle are changing over time. Remember that the volume of a cylinder is given by

$$V = \pi r^2 l$$

where r is the radius of the cylinder and l is the length of the cylinder.

At time $t = 3$ seconds, you are told that the radius is changing at 3 millimeters per second, and that the length is changing at 2 millimeters per second. You are further told that the cylinder is 60 millimeters long with radius 10 millimeters.

Given this information, how quickly is the volume of the cylindrical popsicle changing, in units of cubic millimeters per second?

Exercise 14.2.2. The temperature T of an ideal gas depends on the pressure P and volume V of the gas as follows:

$$PV = kT$$

where k is some number. Assume that the air inside a balloon is an ideal gas. You are being told that the balloon is increasing in volume at 2 cubic centimeters per second, and decreasing in pressure at 3 pascals per second. You are further told that the balloon has a volume of 30 cubic centimeters at time $t = 2$ seconds, and that the air in the volume has pressure exactly 1 pascal at $t = 2$ seconds.

Then, at time $t = 2$, how quickly is the quantity kT changing? Give me your answers in pascal-cubic-centimeters-per-second.

14.3 For next time

For next time, you should be able to do all the exercises in Section 14.1.