Lecture 13

Mean Value Theorem, II

13.1 Some logic

We're going to talk about logic for a second.

13.1.1 Implications

Here are two statements:

1. If R is a rectangle, then R is a square.

2. If R is a square, then R is a rectangle.

Only one of these statements is true!

Most meaningful logical statements can be written in the form

If BLAH, then BLERG

where BLAH and BLERG are statements with some truth value. To be fancy, we often write instead

If P, then Q

where P and Q are fancier stand-ins for BLAH and BLERG. For example, we obtained the above two statements by taking

1. P = "R is a rectangle" and Q = "R is a square."

2. Q = "R is a rectangle" and P = "R is a square."

To be lazy, instead of writing "If P then Q," we may sometimes write

$$P \implies Q.$$

Warning 13.1.1. The following all mean the exact same thing:

- If P, then Q.
- If P is true, then Q is true.
- Given P, we know Q holds.
- P implies Q.
- If we take P as a hypothesis, Q follows as a conclusion.
- In the situation of P, Q is true.
- Suppose P. Then Q is true.
- Whenever P is true, Q is true, too.
- $P \implies Q$.

Remark 13.1.2. Most useful mathematical statements are of the form $P \implies Q$ (or, equivalently, any of the above re-phrasings of this implication). For example,

1. If $f(x) = e^x$, then $f'(x) = e^x$.
2. If a number is even, then it is divisible by 2.

However, as you can see from Warning 13.1.1, the statement $P \implies Q$ may be rephrased in many different ways. This is nice as a matter of poetry and of language; but it can be difficult for a student to pick up on the meanings of statements.

13.1.2 Converses

Given a statement of the form $P \implies Q$, the *converse* is the statement $Q \implies P$.

Example 13.1.3. The converse of "If R is a rectangle, then R is a square" is the statement "If R is a square, then R is a rectangle."

Consider the statement "Every square is a rectangle." The converse statement is "Every rectangle is a square ."

The converse of the converse of a statement is the original statement.

Warning 13.1.4. As you can see from the example, even if a statement is true, the converse statement may be false!

13.1.3 Contrapositives

The *contrapositive* of the statement $P \implies Q$ is the statement

$$\text{not } Q \implies \text{not } P.$$

Warning 13.1.5. "Not Q" is an instance of mathematicians being lazy. For example, if Q is the statement

"R is a rectangle"

then not Q would be the statement

"R is not a rectangle"

In fact, the shorthand "not Q" would be better translated as "Q is not true."

In other words, the contrapositive to $P \implies Q$ is "Q is not true $\implies$ P is not true."

Example 13.1.6.

1. Consider the statement "If R is a square, then R is a rectangle." The contrapositive of this statement is "If R is not a rectangle, then R is not a square."

2. Consider the statement "If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$." The contrapositive of this statement is "If $f'(x)$ does not equal $\cos(x)$, then $f(x)$ does not equal $\sin(x)$."

3. Consider the statement "If $x = 3$, then $x \geq 2$." The contrapositive is "If $x < 2$, then $x \neq 3$."

The greatest fact in logic is: A statement is true whenever its contrapositive is true. A contrapositive is true whenever the original statement is.

Put another way, **a statement is always logically equivalent to its contrapositive.**

Exercise 13.1.7. Find the contrapositives to the following statements:

(a) If f is not constant, then there is some point x at which $f'(x) \neq 0$.

(b) If $f(x)$ is the function e^x, then f always outputs a positive number.

(c) If x is an inflection point of f, then $f''(x) = 0$.

13.2 Application: Functions with the same derivative

Now we are going to prove a fantastic fact.

Proposition 13.2.1. If two functions f and g have the same derivative, then $f - g$ is a constant function.

A "Proposition" is a statement that is true, that requires proof, that is helpful to your understanding, and is not too difficult to prove.

Every Proposition I state for you is true, and you may use it freely. But I want you to not only know true facts; I want you to know why they are true. So let's see a proof of the proposition:

Proof. We first compute a derivative:

$$(f - g)' = f' - g' \tag{13.2.1}$$
$$= 0. \tag{13.2.2}$$

This first equality is the addition rule for derivatives. The next equality uses the fact that f and g have the same derivative—that is, $f' = g'$.

This computation shows us that $f - g$ is a function whose derivative is 0.

Now we apply the contrapositive of a previous fact: If a function's derivative is 0, then it is constant! Applying the contrapositive to the function $f - g$, we conclude that $f - g$ is constant. QED. □

Example 13.2.2. Let's find all functions whose derivative is $\cos(x)$.

We know one such function: $f(x) = \sin(x)$. If g is any other function with $g'(x) = \cos(x)$, we know that $f - g$ is constant; in other words,

$$g(x) = \sin(x) + C$$

for some real number C.

Examples of such $g(x)$ would be functions like

$$g(x) = \sin(x) + 10, \qquad g(x) = \sin(x), \qquad g(x) = \sin(x) - e, \qquad \text{etc.}$$

And *any* function with $g'(x) = \cos(x)$ is a function obtained by adding a single number to $\sin(x)$.

13.3 Application: Functions with positive derivatives grow

Here is another intuitive, but very useful fact:

Proposition 13.3.1. Suppose that f is differentiable, and that $f'(x)$ is always positive. Then whenever $b > a$, we have that $f(b) > f(a)$.

In other words, if f has positive derivative, then f is increasing.

Proof. Suppose that $f(b) \leq f(a)$ for some $b > a$. Then

$$\frac{f(b) - f(a)}{b - a} \leq 0.$$

This is because the denominator is positive, while the numerator is 0 or negative.

By the mean value theorem, there would then be some c in $[a, b]$ such that $f'(c) \leq 0$. This violates our hypothesis that f' is always positive.

Thus, it could not be true that $f(b) \leq f(a)$ for some $b > a$. In other words, for all $b > a$, we have that $f(b) > f(a)$. QED. □

Example 13.3.2. Let $f(x) = e^x - 3$. The derivative of f is always positive because e^{anything} is positive. The proposition above tells us that f is always increasing.

13.4 Application: Comparing values by comparing derivatives

Here is another intuitive, but very useful fact:

Proposition 13.4.1. Suppose that f and g are two differentiable functions. Suppose also that a is a number satisfying

1. $f(a) \geq g(a)$, and

2. For all $b > a$, we have that $f'(b) > g'(b)$.

Then $f(b) > g(b)$ for all $b > a$.

In words, the proposition says that if f at least as large as g at a, and if f has bigger derivatives than g from a onward, then f always remains bigger than g.

Proof. By (2), we can conclude that $f'(b) - g'(b) > 0$ for all $g > a$.

Applying Proposition 13.3.1 to the function $f - g$, we can thus conclude that

$$f(b) - g(b) > f(a) - g(a)$$

for all $b > a$. The righthand side is ≥ 0 by (1), so

$$f(b) - g(b) > 0$$

for all $b > a$. In other words,

$$f(b) > g(b) \text{ for all } b > a.$$

QED. □

Example 13.4.2. Let's compare the functions $f(x) = e^x$ and $g(x) = x + 1$. These functions agree at $a = 0$. But $f'(b) = e^b > 1$ whenever $b > 0$. Meanwhile, $g' = 1$. So f' has bigger derivative when $b > 0$. In other words, $f(b)$ will always be larger than $g(b)$ for positive choices of b.

Isn't that amazing? Even without graphing these two functions, we know that the graph of one will always have a bigger y-coordinate than the other, so long as we're studying the graph for positive x-coordinates!

13.5 For next time

For next time, I'd like you to tell me the converse, and the contrapositive, of the following "if then" statements.

(a) If it is raining, then there is a cloud overhead.

(b) If R is a rectangle, then R has four sides.

(c) If f is continuous on the interval $[a, b]$, then f attains a maximal value at some point in the interval $[a, b]$.[1]

(d) If you have two feet, then you need two shoes.

[1] Note that you do *not* need to know the meanings of words like "continuous" and "maximal value" to write the converse or contrapositive of this statement.