Lecture 9

The product rule and quotient rule

Today, we'll see how to take the derivative of a *product* of two functions.

9.1 The Leibniz rule aka product rule

The Leibniz rule (also known as the product rule). If f and g have derivatives at x, then so does the product fg, and the derivative is computed as follows:

$$\left(\frac{d}{dx}(f \cdot g)\right)(x) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)).$$

Written another way,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Remark 9.1.1. It's up to you which name you prefer for this rule—Leibniz rule or product rule—just be aware that both names are in common use. (So you know what other people are talking about if they mention it.)

Also, remember that in math, the word *product* means the multiplication of something; more specifically, something is called a product if it is being viewed as the multiplication of two things.

For example, 15 is the product of 3 and 5. You can also multiply functions, so that $3x\cos(x)$ is the product of $3x$ and $\cos(x)$.

Example 9.1.2. Let's compute the derivative of $x^2\sin(x)$. We have

$$\frac{d}{dx}(x^2\sin(x)) = \left(\frac{d}{dx}(x^2)\right) \cdot \sin(x) + x^2 \cdot \left(\frac{d}{dx}(\sin(x)\right) \tag{9.1.1}$$

$$= 2x \cdot \sin(x) + x^2\cos(x). \tag{9.1.2}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^2 (the power law from last class) and of the derivative of sin (from this class).

Example 9.1.3. Let's compute the derivative of $x^3 \cos(x)$. We have

$$\frac{d}{dx}(x^3 \cos(x)) = \left(\frac{d}{dx}(x^3)\right) \cdot \cos(x) + x^3 \cdot \left(\frac{d}{dx}(\cos(x)\right) \tag{9.1.3}$$
$$= 3x^2 \cdot \cos(x) - x^3 \sin(x). \tag{9.1.4}$$

The first line is the Leibniz rule, and the next equality follows from our knowledge of the derivative of x^3 (the power law from last class) and of the derivative of cos (from this class).

Example 9.1.4. Let's compute the derivative of $(3x^2+x)(x-3)$. This won't involve any sin or cos.

There are two ways to do this. One way to do this is by using the Leibniz rule:

$$\frac{d}{dx}\big((3x^2+x)(x-3)\big) = \frac{d}{dx}(3x^2+x) \cdot (x-3) + (3x^2+x) \cdot \frac{d}{dx}(x-3) \tag{9.1.5}$$
$$= (3 \cdot 2x + 1) \cdot (x-3) + (3x^2+x) \cdot 1 \tag{9.1.6}$$
$$= (6x+1) \cdot (x-3) + (3x^2+x)1 \tag{9.1.7}$$
$$= 6x^2 - 17x - 3 + 3x^2 + x \tag{9.1.8}$$
$$= 9x^2 - 16x - 3. \tag{9.1.9}$$

Another way is to first multiply the factors together, and then use the addition and power rules:

$$\frac{d}{dx}\big((3x^2+x)(x-3)\big) = \frac{d}{dx}\big(3x^3 - 9x^2 + x^2 - 3x\big) \tag{9.1.10}$$
$$= \frac{d}{dx}\big(3x^3 - 8x^2 - 3x\big) \tag{9.1.11}$$
$$= 3 \cdot 3x^2 - 2 \cdot 8x - 3 \tag{9.1.12}$$
$$= 9x^2 - 16x - 3. \tag{9.1.13}$$

You will get a lot of practice with the product rule in lab tomorrow.

9.2 Quotient Rule

Recall that in math, the word *quotient* refers to something obtained by division. For example, 3 is the quotient of 15 by 5. Likewise, the function

$$\frac{3x}{\cos x}$$

is the quotient of $3x$ by $\cos x$.

First, let's suppose $g(x)$ is a function whose derivative we understand. Let's try to figure out the derivative of the function

$$\frac{1}{g(x)}.$$

Example 9.2.1. We know the derivative of x^3+3. Can we compute the derivative of

$$\frac{1}{x^3+3}?$$

Here is a fact:

Lemma 9.2.2. Whenever $g(x) \neq 0$ and g has a derivative at x, we have:

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-\frac{dg}{dx}(x)}{g(x)^2}.$$

Put another way,

$$\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}.$$

Put yet another way,

$$\left(\frac{1}{g}\right)'(x) = \frac{-g'(x)}{g(x)^2}.$$

Here is a proof of this fact:

Proof. Let's begin by noticing that

$$1 = g(x) \cdot \frac{1}{g(x)} \qquad \text{(whenever } g(x) \neq 0\text{)}.$$

Because the function on the right is equal to the function on the left (they are both constant functions), their derivatives will be equal. Taking the derivatives of both sides, we have:

$$0 = \frac{d}{dx}(g(x) \cdot \frac{1}{g(x)}).$$

Using the product rule on the righthand side, we find:

$$0 = g'(x) \cdot \frac{1}{g(x)} + g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Moving a term over to the left, we find

$$-g'(x) \cdot \frac{1}{g(x)} = g(x) \cdot \frac{d}{dx}\frac{1}{g(x)}.$$

Dividing by $g(x)$ and simplifying, we find:

$$\frac{-g'(x)}{g(x)^2} = \frac{d}{dx}\frac{1}{g(x)} \qquad \text{when } g(x) \neq 0.$$

This proves the lemma! □

Example 9.2.3. Now let's try tackling the question from Example 1.2.1. Let $g(x) = x^3 + 3$. Then we know $g' = 3x^2$. So the formula from the Lemma tells us:

$$\frac{d}{dx}(\frac{1}{g(x)}) = \frac{-g'(x)}{g(x)^2} \tag{9.2.1}$$

$$= \frac{-(3x^2)}{(x^3+3)^2} \tag{9.2.2}$$

$$= \frac{-3x^2}{(x^3+3)^2}. \tag{9.2.3}$$

I won't simplify this fraction any further; though you could multiply out the bottom if you like.

Now, we will finally know how to take derivatives of quotients:

Theorem 9.2.4 (The quotient rule.)**.** Whenever f and g are differentiable at x, and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}.$$

Put another way,

$$\frac{d}{dx}\left(\frac{f}{g}\right)(x) = \frac{\frac{df}{dx}(x)g(x) - \frac{dg}{dx}(x)f(x)}{g(x)^2}.$$

Written yet another way, the quotient rule is

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

In case you're curious, here is the proof of the quotient rule:

Proof of the quotient rule. Let's note that

$$\frac{f}{g} = f \cdot \frac{1}{g}.$$

So we can compute

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) && (9.2.4)\\
&= (\frac{d}{dx}f)(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{d}{dx}\left(\frac{1}{g(x)}\right) && (9.2.5)\\
&= \frac{\frac{df}{dx}(x)}{g(x)} + f(x) \cdot \left(\frac{-\frac{dg}{dx}(x)}{g(x)^2}\right) && (9.2.6)\\
&= \frac{g(x)\frac{df}{dx}(x)}{g(x)^2} + \frac{-f(x)\frac{dg}{dx}(x)}{g(x)^2} && (9.2.7)\\
&= \frac{\frac{df}{dx}(x)g(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2}. && (9.2.8)
\end{aligned}$$

The most important thing to note is that to conclude the equality in (1.2.6), we used Lemma 1.2.2 from above.

The beginning and the end of this string of inequalities is exactly what the quotient rule says. □

Remark 9.2.5. If the notation in the above proof is unappealing, here is a proof using only the "prime" notation:

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' \tag{9.2.9}$$

$$= f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' \tag{9.2.10}$$

$$= \frac{f'}{g} + f \cdot \left(\frac{-g'}{g^2}\right) \tag{9.2.11}$$

$$= \frac{gf'}{g^2} + \frac{-fg'}{g^2} \tag{9.2.12}$$

$$= \frac{f'g - fg'}{g^2}. \tag{9.2.13}$$

Again, the equality in (1.2.11), follows from Lemma 1.2.2 from above.

Warning 9.2.6. The hardest part about the quotient rule is remembering the order of things in the numerator:

$$f'g - fg'.$$

Note that the positive term is $f'g$, and the negative term is fg'. Some people prefer to remember the numerator as

$$f'g - g'f,$$

which is the same thing.

Do *not* make the mistake of writing something like $g'f - f'g$ in the numerator. This is the wrong answer.

Example 9.2.7 (The derivative of tangent)**.** Let's compute the derivative of

$$\frac{\sin(x)}{\cos(x)}.$$

(This is known, of course, as the *tangent* function.) Follow along:

$$\left(\frac{\sin}{\cos}\right)' = \frac{\sin' \cdot \cos - \sin \cdot \cos'}{\cos^2} \tag{9.2.14}$$

$$= \frac{\cos \cdot \cos - \sin \cdot (-\sin)}{\cos^2} \tag{9.2.15}$$

$$= \frac{\cos^2 + \sin^2}{\cos^2} \tag{9.2.16}$$

$$= \frac{1}{\cos^2} \tag{9.2.17}$$

$$= \sec^2. \tag{9.2.18}$$

The first equality is using the quotient rule. The equality (1.2.17) follows from the identity $\sin^2 + \cos^2 = 1$. (You will be expected to know this identity; it's from trigonometry!) The last equality (1.2.18) follows from the definition of secant: $\sec = 1/\cos$.

We have proven:

$$\frac{d}{dx}\tan = \sec^2.$$

Equivalently,

$$\tan'(x) = \sec(x)^2.$$

Example 9.2.8. Find the derivative of

$$\frac{x^2-3}{x^3+1}.$$

We use the quotient rule:

$$\left(\frac{x^2-3}{x^3+1}\right)' = \frac{(x^2-3)'\cdot(x^3+1)-(x^2-3)\cdot(x^3+1)'}{(x^3+1)^2} \tag{9.2.19}$$
$$= \frac{2x\cdot(x^3+1)-(x^2-3)\cdot 3x^2}{(x^3+1)^2} \tag{9.2.20}$$
$$= \frac{2x^4+2x-3x^4+9x^2}{(x^3+1)^2} \tag{9.2.21}$$
$$= \frac{-x^4+9x^2+2x}{(x^3+1)^2}. \tag{9.2.22}$$

9.3 For next lecture

For next lecture, I expect you to be able to do any of the following exercises:

Exercise 9.3.1. Compute the derivatives of the following functions:

1. $-\cos(x)$
2. $x-\cos(x)$
3. $-x\cos(x)$
4. $-x^2\cos(x)$.

5. $\sin(x)^2$.

6. $\sin(x)\cos(x)$.

7. $x^3 + 3x - 2$.

8. $(x-3)(x-2)$.

9. $(x^2-1)(3x-1)$.

Exercise 9.3.2. What is the slope of the tangent line to the graph of $f(x) = x\cos(x)$ at $x = \pi/4$?