Lecture 6

The chain rule

Here are the summaries of the rules/laws we know for derivatives so far:

Rule/Law	For derivatives
Constants	$(C)' = 0.$
Scaling	$(mf)' = mf'$
Sums	$(f' + g') = f' + g'$
Powers	$(x^n)' = nx^{n-1}.$
Composition	$(f \circ g)' =???$
Products	$(fg)' =???$
Quotients	$(f/g)' =???$

Today, we are going to practice taking derivatives of *compositions*. The rule we use to compute derivatives of composition is called the *chain rule*.

6.1 Review of compositions

The hardest part of applying the chain rule, for most calculus students, is actually understanding what a composition of functions is.

Remember that functions take inputs and produce outputs. A *composition* happens when a second function uses a first function's output as the second function's input. If you like, a composition is like a relay race in track and field. The first function passes a *number* onto the second function (instead of a baton).

When f is a function, and g is another function, we write

$$g \circ f$$

for the composition. When we evaluate $g \circ f$ at a number x, we have:

$$(g \circ f)(x) = g(f(x)).$$

The righthand side, in words, says: Apply f to x, and whatever $f(x)$ is, plug it into g.

Example 6.1.1. Let $f(x) = x + 2$ and $g(x) = x^2$. Then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= g(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 2. \end{aligned} \tag{6.1.1}$$

Example 6.1.2. Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= g(\sin(x)\cos(x)) \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

If you like, this last expression could also be written as

$$\sin(x)^2\cos(x)^2 + 3\sin(x)\cos(x) + 2 \qquad \text{or} \qquad \sin^2(x)\cos^2(x) + 3\sin(x)\cos(x) + 2$$

You can also try to compute the "outside function" first.

Example 6.1.3. Let $f(x) = \sin(x)\cos(x)$ and $g(x) = x^2 + 3x + 2$. then

$$\begin{aligned} (g \circ f)(x) = &= g(f(x)) \\ &= (f(x))^2 + 3f(x) + 2 \\ &= (\sin(x)\cos(x))^2 + 3\sin(x)\cos(x) + 2. \end{aligned}$$

6.2 The Chain Rule

Theorem 6.2.1 (Chain rule)**.** Suppose that g is differentiable at x, and that f is differentiable at $g(x)$. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Put another way,

$$\frac{d(f \circ g)}{dx}(x) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$$

I want to emphasize in words what the chain rule says: If you want to compute the derivative of $f \circ g$ at x, then you must compute two things:

1. The derivative of f **at g(x)**, and
2. The derivative of g at x.

The product of these two numbers gives the derivative of $f \circ g$ at x.

Using the chain rule, you can find the derivative of functions like

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

6.3 Identifying compositions

For many calculus students, the hardest part about taking derivatives is knowing whether we can use the chain rule in a particular situation. Why is this so hard? Well, in past classes, you've learned how to compose two functions, but you haven't learned to recognize whether a given function *arises* as a composition. Moreover, to use the chain rule, you need to be able to recognize the functions that are being composed.

Example 6.3.1. Let's write each of the functions below as a composition $g \circ f$. Importantly, let's identify the functions g and g.

(a) $(\sin(x))^3$

(b) $\sin(x^3)$

(c) $\cos(x^4 + 3x^3 - 2)$.

Solution:

(a) What this expression tells us to do is to *first* evaluate $\sin(x)$, and then cube the result. So the first function is $f(x) = \sin(x)$ and the second, or "outside" function is $g(x) = x^3$.

(b) This expression tells us to first cube x, and then take sin of the result. So the first function is $f(x) = x^3$, and the second, or "outside" function is $g(x) = \sin(x)$.

(c) This expression tells us to take a number x, and first evaluate $x^4 + 3x^3 - 2$, and then take cos of the result. So $f(x) = x^4 + 3x^3 - 2$, and $g(x) = \cos(x)$.

You should check in all three examples that $(g \circ f)(x)$ indeed gives rise to the original expression.

6.4 Applying the chain rule

Now let's apply the chain rule.

Example 6.4.1. Find the derivative of $\sin(x^2 + 5)$.

Solution. We must first recognize that we behold a composition of two functions: On the outside is sin, while the inside is $x^2 + 5$. Hence we can use the chain rule.

$$\frac{d}{dx}(\sin(x^2+5)) = (\frac{d}{dx}\sin)(x^2+5) \cdot \frac{d}{dx}(x^2+5).$$

This is a product of two factors: The first factor, on the left, is the derivative of sin, evaluated at $x^2 + 5$. The second factor, on the right, is the derivative of $x^2 + 5$ (evaluated at x).

Because we know $\frac{d}{dx}\sin = cos$, and that $\frac{d}{dx}(x^2 + 5) = 2x$, we conclude:

$$\frac{d}{dx}(\sin(x^2+5)) = \cos(x^2+5) \cdot 2x.$$

Or, in more palatable notation,

$$\frac{d}{dx}(\sin(x^2+5)) = 2x\cos(x^2+5).$$

Exercise 6.4.2. Find the derivatives of the following functions:

1. $(\cos(x) + \sin(x))^3$
2. $\cos(\sin(x))$
3. $\cos(2x^4)$.

Exercise 6.4.3. We do not yet know how to take derivatives of a function like $h(x) = x^{1/3}$. *However*, we do know that if $g(x) = x^3$, then $g(h(x)) = x$.

Using this, and the chain rule, can you find a formula for $h'(x)$? That is, can you compute the derivative of $x^{1/3}$?

6.5 For next time: Exponentials, logarithms, and e (A primer)

Remark 6.5.1. If you are already comfortable with functions like e^x and $\ln x$, and how they relate to functions like 2^x and $\log_2 x$, you can focus on Section 6.5.6.

For next time, you'll need to be prepared to use exponentials and logarithms.

Consider the function $f(x) = 4^x$. You have seen this in precalculus. In fact, you probably knew that

$$4^0 = 1, \qquad 4^1 = 4, \qquad 4^2 = 4 \times 4 = 16, \qquad 4^3 = 4 \times 4 \times 4 = 64,$$

et cetera, back in high school. The cool fact is that even if x is not an integer, 4^x is a number that makes sense.

Example 6.5.2. Here are some examples:

1. It makes sense to raise something to a negative power:
$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}.$$
More generally, we have that
$$4^{-n} = \frac{1}{4^n}.$$

2. It makes sense to raise something to a fraction:
$$4^{\frac{1}{3}} = \sqrt[3]{4} \text{ is the cube root of 4.}$$
More generally, we have that
$$4^{1/n} = \sqrt[n]{4}$$
is the nth root of 4. This root is the unique positive number so that its nth power is equal to 4.

3. Another fraction example:
$$4^{\frac{2}{3}} = \sqrt[3]{4^2}.$$
(Note that this also equals $(\sqrt[3]{4})^2$.) Put into English, this means that $4^{2/3}$ is the cube root of 4^2, or the square of the cube root of 4. More generally,
$$4^{\frac{a}{b}} = \sqrt[b]{4^a} = (\sqrt[b]{4})^a.$$

6.5.1 Exponent laws!

Let's see why the above statements are true.

You may have learned about the exponent laws in precalculus or back in high school. One of these laws says:

$$4^{a+b} = 4^a \cdot 4^b$$

That is, exponentiation takes "addition" to "multiplication." For example, $4^7 = 4^{2+5} = 4^2 \cdot 4^5$. Another law says:

$$4^{a \cdot b} = (4^a)^b = (4^b)^a.$$

This means that exponentiation takes "multiplication" to "powers." For example, $4^{21} = 4^{3 \cdot 7} = (4^3)^7$. Also, we have that $4^{21} = (4^7)^3$.

Example 6.5.3. Let's verify that the exponent laws are consistent with our knowledge of math. We have:

$$4^{2+3} = 4^5 = 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4 \times 4) = 4^2 \times 4^3.$$

So indeed, $4^{2+3} = 4^2 \cdot 4^3$.

We also have:

$$4^{2 \cdot 3} = 4^6 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 = (4 \times 4) \times (4 \times 4) \times (4 \times 4) = (4 \times 4)^3 = (4^2)^3.$$

Remark 6.5.4 (Reminder)**.** Let me also remind you that *anything* to the 0th power is equal to 1. For example, $5^0 = 1$. Likewise, $\pi^0 = 1$.

And, anything to the 1st power is that anything again. For example, $5^1 = 5$.

Remark 6.5.5 (The reasoning for fractional and negative powers)**.** Knowing these laws is how you *create* the definitions for things like 4^{-3} and $4^{1/5}$. Indeed, if you know what 4^3 is, and if you desire the law $4^{3+(-3)} = 4^3 \cdot 4^{-3}$ to be true, you *must* conclude that 4^{-3} is equal to $1/4^3$. For example,

$$4^3 \cdot 4^{-3} 4^{3+(-3)} = 4^0 = 1.$$

Dividing both sides by 4^3, we see

$$4^{-3} = \frac{1}{4^3}.$$

Likewise, the other law of exponent tells us

$$5 = 5^1 = 5^{\frac{1}{2}\cdot 2} = (5^{\frac{1}{2}})^2.$$

Taking the square root of both sides, we find

$$\sqrt{5} = 5^{\frac{1}{2}}.$$

There's nothing special about the number 5 here; anything to the $\frac{1}{2}$ power is the square root of that anything. Likewise, anything to the $\frac{1}{3}$ power is the cube root.

6.5.2 The number e

The number e is called Euler's constant sometimes, but it's usually just called e. (Eeee!) In your previous math classes, you probably didn't have too much reason to care about this deeply, except that it has some interesting roots in banking. However, you will see why e is important in calculus.

For now, let me just say that e is an irrational number, and here are the first few digits of its decimal expantion:

$$2.71828182845904523536028747135 2\ldots \tag{6.5.1}$$

We will soon be dealing with the function $f(x) = e^x$. You should think of this function as behaving very much like $f(x) = 4^x$. For example, we have that

$$e^0 = 1, \qquad e^1 = e, \qquad e^2 = e \times e \approx 7.38905609\ldots.$$

(We compute e^2 using a computer or calculator; if we have a lot of time at the end of this course, we'll see *how* a computer does this!)

6.5.3 The logarithm

The logarithm base n of a number x is written

$$\log_n x.$$

The number $\log_n x$ is the number you need to raise n to in order to obtain a value x. For example,

$$\log_3 9 = 2.$$

This is because 2 is the number such that $3^2 = 9$. As another example,

$$\log_3 81 = 4.$$

(Just try computing 3^4 to see why this is true.)

Put another way, it is always true that

$$3^{\log_3 x} = x.$$

We say that the logarithm base n is the *inverse* function to exponentiation base n. (Put another way, if the output of the logarithm becomes the input of the exponential, the final output is the initial input.)

In fact, it is also true that

$$\log_3(3^x) = x.$$

Exercise 6.5.6. You should be able to compute the following:

(a) $\log_2 8$

(b) $\log_3 243$

(c) $\log_2 \sqrt{2}$

(d) $\log_\pi \pi^3$.

6.5.4 Natural logarithm

Because e is so special[1], we give a special name to the logarithm base e. We define the *natural logarithm*, or the *natural log*, to be the logarithm base e, and we denote it as follows:

$$\ln$$

So for example,

$$\ln e = 1, \qquad \ln(e^3) = 3.$$

In general ln of a nice integer looks crazy; for example,

$$\ln 2 = 0.69314718056\ldots$$

so if you like integers and rational numbers, ln is not your best friend. But it will become a better friend as we realize how important ln and e are in calculus—in fact, it is probably one of the most convincing pieces of evidence that crazy, *transcendental numbers* like e have a place in our mathematical universe.[2]

[1] We will see why in the coming lectures

[2] There is another transcendental number, π, that is obviously very important to mathematics. If you don't know what a transcendental number is, don't worry; they are a special kind of irrational number.

In calculus, it will be very useful to know how to convert expressions like

$$5^{\text{some power}}$$

into expressions base e; that is, into expressions like

$$e^{\text{some other power}}.$$

Example 6.5.7. Let us convert 5^3 into an exponent with base e. Here is our work:

$$\begin{aligned} 5^3 &= (e^{\ln 5})^3 && (6.5.2)\\ &= e^{(\ln 5)\cdot 3} && (6.5.3)\\ &= e^{3\ln 5}. && (6.5.4) \end{aligned}$$

The first equality is using the definition of logarithm base e. (Note that we don't need to know how to calculate $\ln 5$, but we know that it exists as a number, so we just use it.) The next equality follows from an exponent law: Exponentiation exchanges multiplication of powers with iterated powers. The last line is just re-writing the same expression in a nicer way.

6.5.5 Exponentials for non-rational powers

Now, you may not have thought deeply about how to calculate something like 4^x when x is, say, an irrational number. In this class, you only need to know this *can* be done, and not *how* to do it.

So you only need to read this section if you're curious about how something like 4^π is computed. I'll illustrate by example.

Example 6.5.8. For example, how would you compute 4^π? It's a three-step process.

First, we choose a collection of numbers that approximates π really well. For example, we could choose

$$3, \qquad 3.1, \qquad 3.14, \qquad 3.141, \qquad 3.1415, \qquad 3.14159, \qquad \ldots$$

and so forth. Note that each of these numbers can be written as a fraction. (For example, 3.14 is equal to 314/100; you can simplify this fraction if you like.) In particular, we *know* how to calculate each of the following numbers:

$$4^3, \qquad 4^{3.1}, \qquad 4^{3.14}, \qquad 4^{3.141}, \qquad 4^{3.1415}, \qquad 4^{3.14159}, \qquad \ldots$$

and so forth.

Calculating all these numbers is the second step. For your edification, here are the answers:

$$64, \qquad 73.516\ldots, \qquad 77.708\ldots, \qquad 77.816\ldots, \qquad 77.870\ldots, \qquad 77.880\ldots,$$

and so forth.

Now, here is the third and most fun/difficult step. We have to *prove* that this collection of numbers "converges" to some number—put another way, that this sequence has a limit.[3] Then we *define* 4^π to be this limit.

6.5.6 For the quiz!

For the quiz, you should be able to simplify the following expression:

(a) $e^{\ln 3}$

(b) $e^{\ln e}$

(c) $e^{\ln x}$

(d) $e^{\ln 1}$

(e) $e^{\ln \pi}$

(f) $\ln(e^3)$

(g) $\ln(e^\pi)$

(h) $\ln(e^x)$

(i) $\ln(e^3 \cdot e^5)$

(j) $\ln(e^3 \cdot e^x)$

(k) $\ln(e^3 \cdot e^{-3})$

[3] Warning: This notion of limit is slightly different from the notion of limit we have discussed before. This is the limit of a *sequence of numbers*, while we have discussed in this class the limit of a *function* at a point.