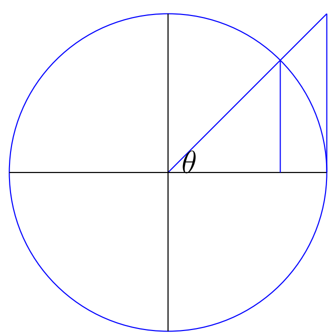
Lecture 5

Derivatives of sine and cosine

Last time we learned how to take derivatives of polynomials.

To day we'll learn how to take derivatives of $\sin(x)$ and $\cos(x)$.

5.1 Warm-up exercise



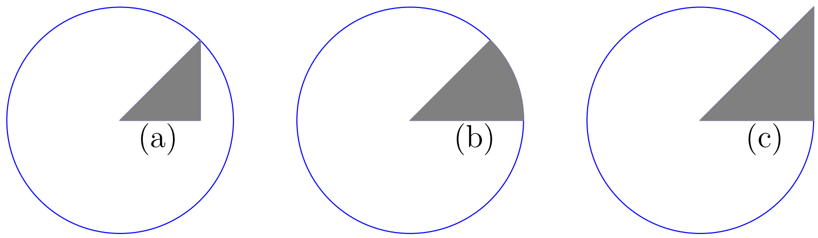
In the above diagram are three shapes associated to an angle of size θ in a circle of **radius one**. There are

(a) A small triangle (inside the circle),

(b) A sector of the circle (formed by the angle θ), and

(c) A larger triangle (not contained in the circle, and whose height is tangent to the circle).

In case it helps, these shapes are drawn as shaded regions below:



Problem. Find the areas of each region, (a), (b), and (c).
We'll come back to this soon.

5.2 Derivatives of sine and cosine

A **theorem** is a statement that is true, that is very powerful, and requires a lot of thinking (or guidance) to see the truth of. As we have seen, the derivative of a function tells us the slopes of tangent lines by understanding certain approximations as a parameter called h goes to zero. One of the amazing facts of life is that we can compute the derivatives of functions like sine and cosine:

Theorem 5.2.1.

$$\frac{d}{dx}(\sin)(x) = \cos(x), \qquad \frac{d}{dx}(\cos)(x) = -\sin(x).$$

Written another way,

$$(\sin)'(x) = \cos(x), \qquad (\cos)'(x) = \sin(x),$$

or

$$\sin' = \cos, \qquad \cos' = -\sin,$$

or

$$\frac{d}{dx}\sin = \cos, \qquad \frac{d}{dx}\cos = -\sin.$$

In words, *the derivative of sine is cosine. The derivative of cosine is* **negative** *sine.*

Example 5.2.2. Let's find the slope of line tangent to the graph of $\sin(x)$ at $x = \pi$. We know from above that

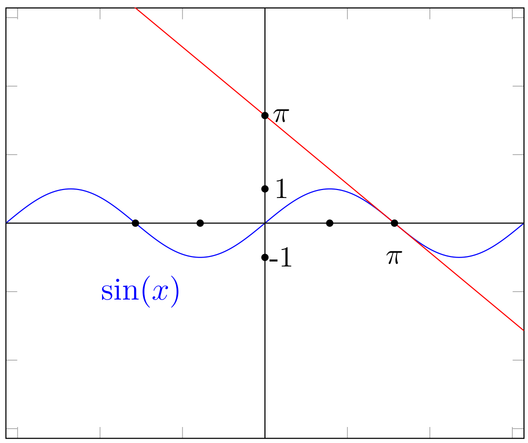
$$(\frac{d}{dx}\sin)(x) = \cos(x),$$

so

$$(\frac{d}{dx}\sin)(\pi) = \cos(\pi).$$

And now we must remember from trigonometry that $\cos(\pi) = -1$.

Here is a picture to confirm that, indeed, the tangent line at $x = \pi$ looks like it has slope -1:



(The tangent line is drawn in red.)

Exercise 5.2.3. Find the derivative of $\cos(x)$ at the following points:

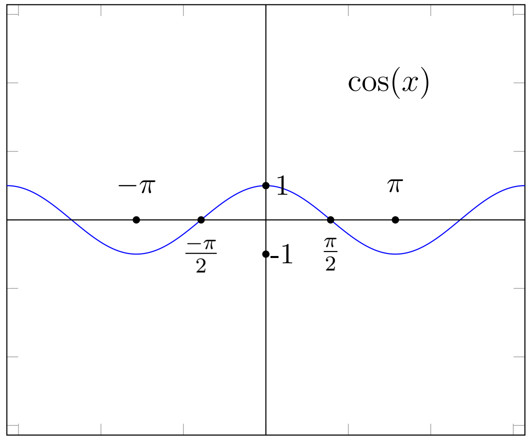
(a) $x = 0$

(b) $x = \pi/2$

(c) $x = \pi$

(d) $x = -\pi$

Do your answers make sense when you look at the graph of $\cos(x)$?



cos(x)
−π
π
1
-1
−π/2
π/2

5.3 The start of proving that $\frac{d}{dx}(\sin x) = \cos x$.

I want to *prove* to you that the derivative of sine is cosine.

Before that, I need to prove two *lemmas*. A lemma is a statement that's a bit tricky to prove, but that you need to prove in order to prove a theorem.

Lemma 5.3.1.

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

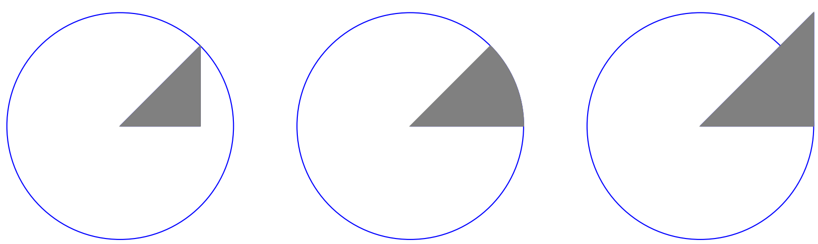
That is, as h approaches zero, the expression $\frac{\sin h}{h}$ approaches 1.

This is an example of a expression where we can't just "plug in $h = 0$"!

Proof of Lemma 2.3.1. Let me first try to convince you of the following:

$$\frac{\sin x \cos x}{2} \leq \frac{x}{2} \leq \frac{\sin x}{2\cos x}. \tag{5.3.1}$$

This inequality is true because these expressions are the areas of the shaded regions below, formed by the angle x:



Note two things: The angle x is *positive* in the pictures above, and also *less than* $\pi/2$—that is, less than 90 degrees. Thus, $\sin x$ is *positive*. So we can divide the inequality in (2.3.5) by $\sin x$ without changing the directions of the inequalities:

$$\frac{\cos x}{2} \leq \frac{x}{2\sin x} \leq \frac{1}{2\cos x}. \tag{5.3.2}$$

Now, note that if I have two numbers a, b satisfying $a \leq b$, then I know that $(1/a) \geq (1/b)$. So, I can "flip the fractions" and flip the inequalities. That is, (2.3.2) implies the following:

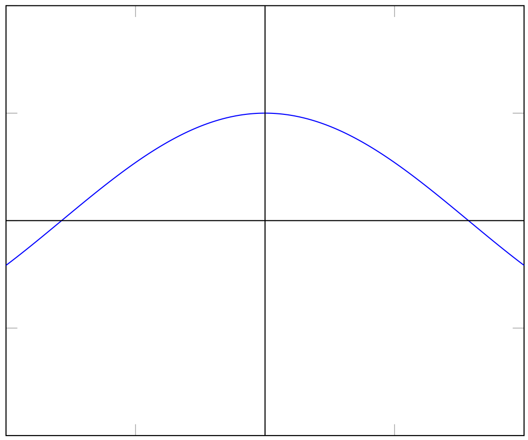
$$\frac{2}{\cos x} \geq \frac{2\sin x}{x} \geq 2\cos x. \tag{5.3.3}$$

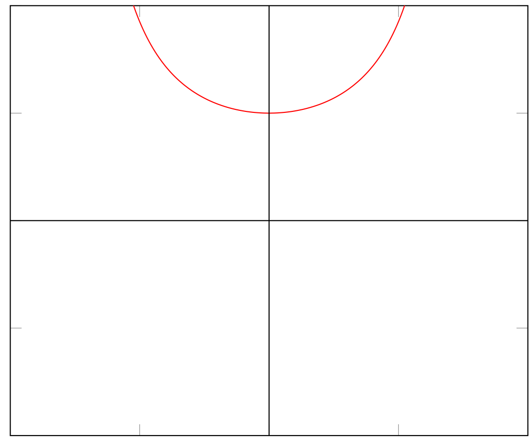
Dividing everything by 2, I obtain

$$\frac{1}{\cos x} \geq \frac{\sin x}{x} \geq \cos x. \tag{5.3.4}$$

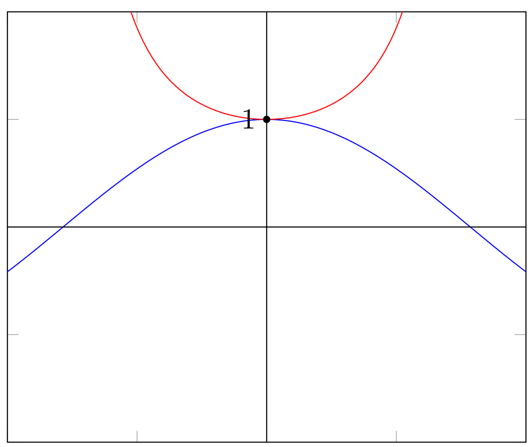
So let's marvel at this for a moment. In the middle is the expression we want to study as x goes to zero. Maybe the expressions on the left and right can help us!

To that end, let's graph what those expressions look like. That is, let's graph the functions $1/\cos(x)$, and $\cos(x)$:



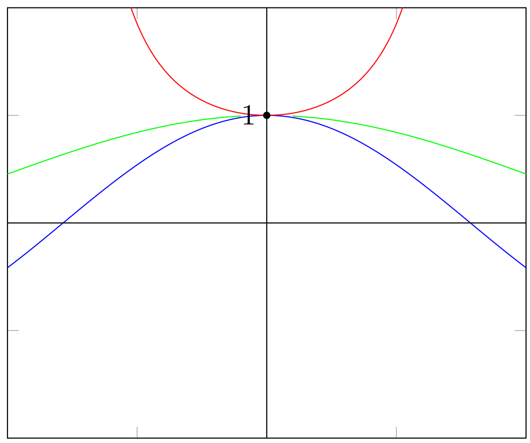


Above, I've drawn the function $\cos(x)$ in blue on the left. (I've only drawn a part of it, the part near $x = 0$.) On the right, I've drawn the function $1/\cos(x)$ in red. (Again, only a part of it.) What the inequality (2.3.4) tells us is that regardless of what x is, the value of $\sin(x)/x$ will be *above* the blue curve, but *below* the red curve. So to help us visualize what value $\sin(x)/x$ could take as x approaches zero, let's put the blue and red curves on the same graph:



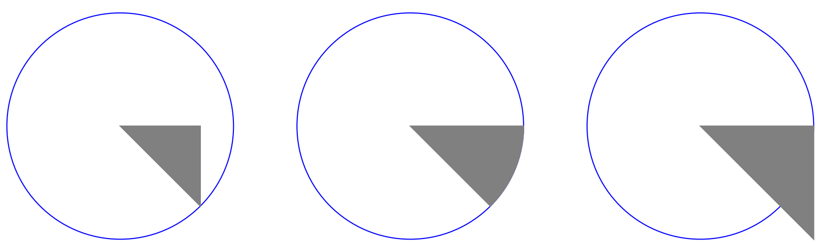
They *touch* each other at $x = 0$, and at height 1! So if $\sin(x)/x$ has to have a graph that's between these blue and red curves (a graph that's define everywhere except at $x = 0$, because we can't divide by 0) it *has* to approach the height of 1 as x approaches 0.

Indeed, just for visual confirmation, here's a graph of $\sin(x)/x$, in green, between the red and blue curves:



I made a big fuss about x being positive above. So actually, we've only studied what happens as we approach 0 *from the right*.

Well, when x is negative (which x must be, if it is approaching 0 from the left), the areas of the shaded regions change.



The areas become

$$\frac{-\sin x \cos x}{2} \leq \frac{-x}{2} \leq \frac{-\sin x}{2\cos x}. \tag{5.3.5}$$

(One way to see this is that areas must always be *positive*, so if x is negative, we must flip the sign of $\sin x$ and x to obtain positive numbers in the inequality.)

Now, dividing by $-\sin x$ (which is a *positive* number when the angle is negative!) we obtain (2.3.2) again, and the rest of the algebraic work we did before carries through. □

You are expected to know this particular limit from now on, and you may use it at will. That is, you are expected to know

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

We haven't yet proven that the derivative of $\sin(x)$ is $\cos(x)$. To complete this proof will be your writing assignment for this week!

Remark 5.3.2. Secretly, I have used what is usually viewed as a fancy trick; something called the *squeeze theorem*. The squeeze theorem says that if I need to compute a limit of some expression, and if I I can "sandwich" the values of that expression, then I can also sandwich the limit value. This is probably intuitive, and I tried to "draw" why this is true using the red and blue curves above. You should know that the squeeze theorem exists, the way that you probably know that Shakespeare exists. But you probably won't need to recite Shakespeare in your everyday life. (Nor the squeeze theorem!)

5.4 Preparation for next time

For next lecture, I expect you to be able to do any of the following exercises:

Exercise 5.4.1. Compute the derivatives of the following functions:

1. $-\cos(x)$
2. $x - \cos(x)$
3. $-x + \cos(x)$
4. $-x^2 + \cos(x)$.
5. $\sin(x) + \cos(x) - x^3$.
6. $3\sin(x) - 2\cos(x) + 3x^2 + 1$.
7. $x^3 + 3x - 2$.

Exercise 5.4.2. What is the slope of the tangent line to the graph of $f(x) = \cos(x)$ at $x = \pi/4$?