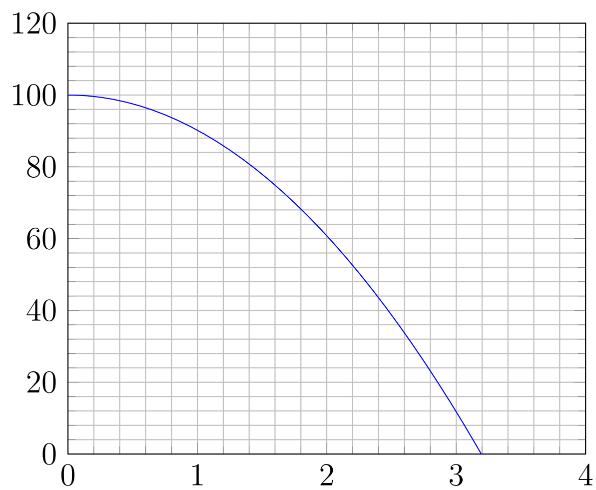
Lecture 2

Tangent lines and secant lines

2.1 Speed of a falling ball

Below is a graph depicting the height $f(t)$ of a ball at time t.



The horizontal axis is in units of seconds (s), and the vertical axis is measured in meters (m). You can interpret the graph as depicting what happens over time when you drop a ball from 100 meters high.

(a) What is the height of the ball at time $t = 2$ seconds?

(b) Suppose your friend tells you the height of the ball after $2+h$ seconds. (Here,

h is some number.) That is, your friend tells you $f(2+h)$. In terms of $f(2)$ and $f(2+h)$, how far does the ball vertically travel between times 2 and $2+h$?

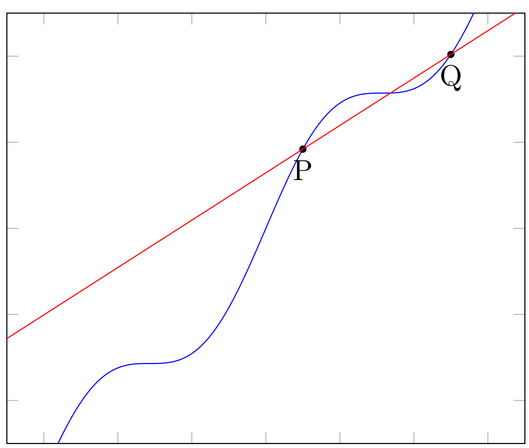
(c) Over that period of time, what is the average speed of the falling ball?

(d) What does this average speed have to do with the line passing through the point $(2, f(2))$ and the point $(2+h, f(2+h))$?

(e) How should you change h if you want to know the speed of the ball *at* time $t=2$?

2.2 Secant lines

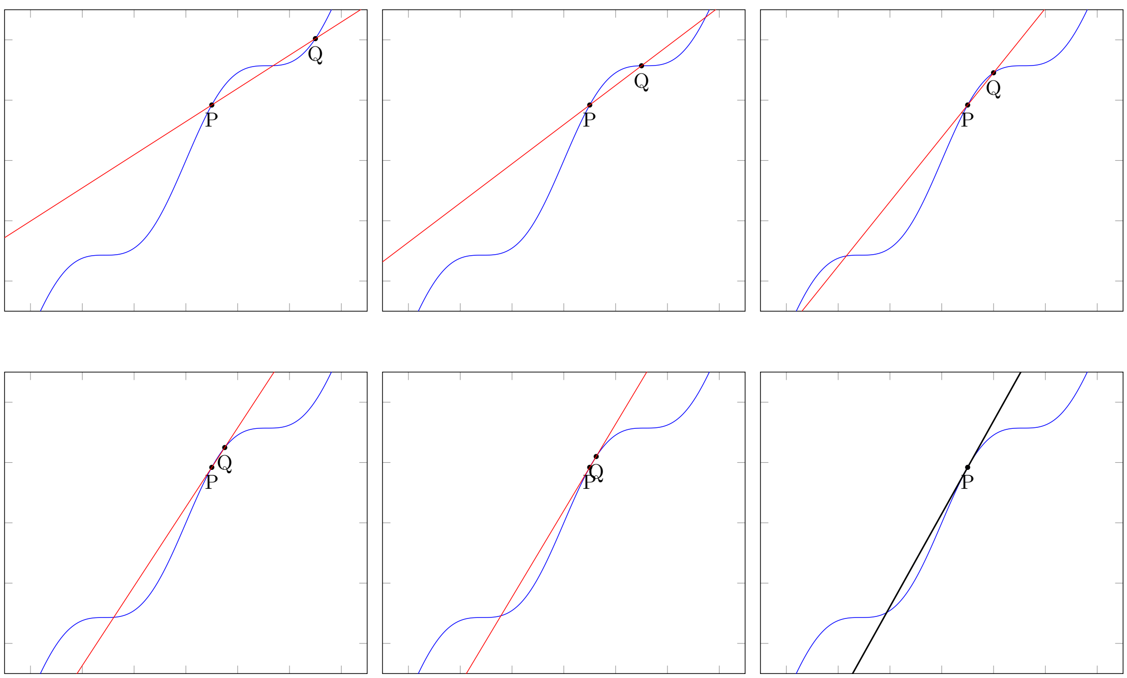
Let f be a function, and consider the graph of f. Choose two points P and Q on this graph. Then the *secant line through* P *and* Q is the line passing through P and Q. (Note that a secant line may pass through the graph at more points than just P and Q.)



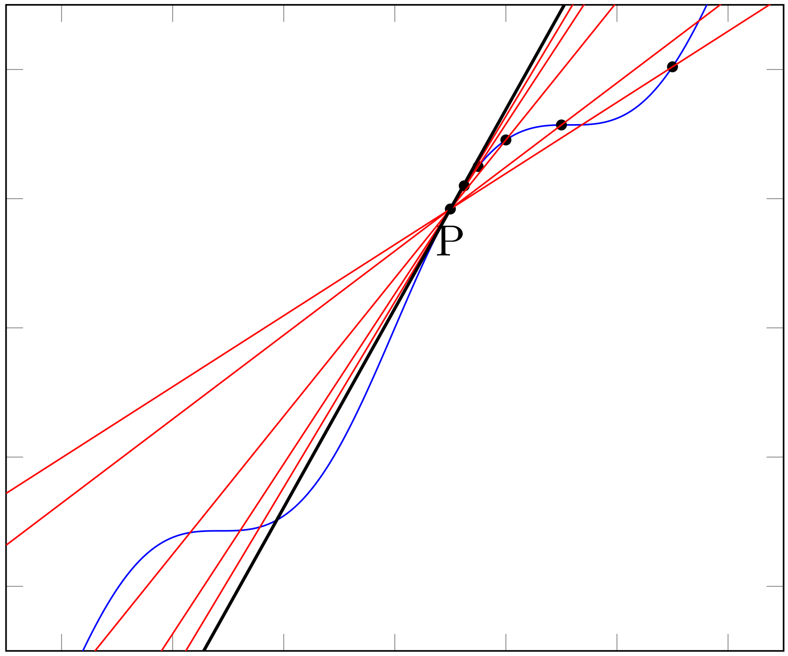
Above, in blue, is the graph of some function f. I chose two points P and Q on this graph, and in red, I drew the line between them. This red line is the secant line passing through P and Q. Note that the secant line *can* pass through more points of the graph of f than just P and Q.

2.3 Tangent lines

Sometimes, if you move the point Q to be closer and closer to P (from either side of P—i.e., from the left or from the right) the secant lines through P and Q approach a line that barely "kisses" the graph of f at P. If such a line exists, we call the line the *tangent line* to the graph of f at P.



Above is a sequence of pictures. In blue is the graph of a function f. We choose a point P, and P doesn't move. But we move a point called Q closer and closer to P. Then the red secant line through P and Q changes (because Q is moving), and the red lines themselves approach the black line in the last image. In the above sequence of pictures, Q is approaching P from the right; it turns out that if we were to make Q approach P from the left instead, the secant lines would still approach the black line. So, following the definition of "tangent line" I just gave, we see that the black line is the tangent line to the graph of f at P.



In this bigger picture, we combine the "movie" of the secant lines (drawn in red) as they approach the tangent line at P (drawn black).

In this lecture, I just wanted you to get to see what secant lines and tangent lines are. The first third of our course will be devoted to you finding the slopes of tangent lines. Finding the slope of the tangent line to f at P is called *taking the derivative* of f at P.

For next lecture

I want you to get practice with expressions that look like the following:

$$\frac{f(x+h)-f(x)}{h}$$

Such expressions are called *difference quotients*. There is a reason these are called difference quotients. First, the numerator measures the *difference* in the value of the function at $x+h$, and the value of the function at x. Then, one divides (i.e., takes the *quotient*) of that difference by the distance between x and $x+h$—this is the h in the denominator.

To evaluate difference quotients concretely, you need to know what the function f is.

Example 2.3.1. If $f(x) = 2x+1$, then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} = \frac{(2(x+h)+1)-(2x+1)}{h} &= \frac{2x+2h+1-2x-1}{h} \\ &= \frac{2h}{h}.\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, the difference quotient equals 2.

Example 2.3.2. If $f(x) = x^2+10$, then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} = \frac{((x+h)^2+10)-(x^2+10)}{h} &= \frac{x^2+2xh+h^2+10-x^2-10}{h} \\ &= \frac{2xh+h^2}{h}\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, it equals $2x+h$.

Example 2.3.3. If $f(x) = x^3$, then

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h} &= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \frac{3x^2h+3xh^2+h^3}{h}\end{aligned}$$

This expression is only defined when $h \neq 0$. When $h \neq 0$, it equals $3x^2+3xh+h^2$.

Once you know what f is, you should be able to tell me what the difference quotient is as an expression of x and of h.

For next class (when you will have another quiz), you should be able to evaluate (and show work for!) the difference quotients of the following functions:

1. $f(x) = 5x + 3$
2. $f(x) = x^2 + x$
3. $f(x) = x^3 + 2$
4. $f(x) = x$