Last time: <u>Categories</u>. 1/29/17

<u>Defn</u> A <u>category</u> $\mathcal{C}$ is the data of

(1) A collection $\mathrm{ob}\,\mathcal{C}$, called the "objects of $\mathcal{C}$"

(2) $\forall X, Y \in \mathrm{ob}\,\mathcal{C}$, a set called $\hom(X, Y)$.

(3) $\forall X, Y, Z \in \mathrm{ob}\,\mathcal{C}$, a function
$\hom(Y, Z) \times \hom(X, Y) \to \hom(X, Z)$.
satisfying

(4) $\forall X \in \mathrm{ob}\,\mathcal{C}$, $\exists\ 1_X = id_X$
st $f \circ 1_X = f$, $1_X \circ g = g$
$\forall f \in \hom(X, Y)$, $\forall g \in \hom(Y, X)$

(5) $\circ$ is associative.

<u>Defn</u> Given $\mathcal{C}, \mathcal{D}$ categories, a <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is the data of

(1) a function $F : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$
$X \longmapsto F(X)$.

(2) $\forall X, Y \in \mathrm{ob}\,\mathcal{C}$, a fn
$F : \hom(X, Y) \longrightarrow \hom(F(X), F(Y))$
$f : X \to Y \qquad F(f) : F(X) \to F(Y)$

st

(3) $F(g \circ f) = F(g) \circ F(f)$

(4) $F(1_X) = 1_{F(X)}$

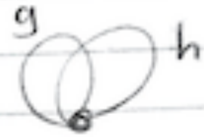
<u>Remark</u> A category can represent/encode a lot of diff. kinds of structures.

(0) $\mathcal{C}$ = groups, rings, sets, fields.

(1) Recall: a <u>poset</u> P is a partially ordered set.
$(P, \leq)$ - $x \leq y$, $y \leq z \Rightarrow x \leq z$
- $x \leq x$
- $x \leq y$, $y \leq x \Rightarrow x = y$

(2) BG

* Recall: BG is a category w/
  - $\text{ob}\, BG = *$
  - $\hom(*,*) = G$



vertices = objects
edges = morphisms

* Define a cat. $\underline{P}$, $\text{ob}\,\underline{P} = P$
$\forall x, y \in P$, $\hom(x,y) := \begin{cases} * & \text{iff } x \le y \\ \emptyset & \text{otherwise.} \end{cases}$

ex. (of functors)

(1) Fix two groups G, H
A functor $BG \to BH$ is the same thing as a group homom $G \to H$.

(2) Let $\mathcal{D} = \text{Set}$
$\mathcal{C} = BG$
A functor $BG \to \mathcal{D}$ is a choice of a set Y w/ a group action.

(3) Let $P = [1] = \{0 \le 1\}$
then $\underbrace{\text{Fun}([1], \mathcal{D})}_{\text{collection of all functors.}} \cong \coprod_{X, Y \in \text{ob}\,\mathcal{D}} \hom(X, Y)$

(4)(i) Fix G a group. Let $G\text{Set}$ be the set where $\text{ob} = \{G \circlearrowright X\} = \{(X, \mu : G \times X \to X)\}$
$\hom((X,\mu),(X',\mu')) = \{G\text{-equiv maps } X \xrightarrow{f} X'\}$
$f(gx) = g(f(x))$ $\forall g, x$.

(ii) Any gp homom $G \xrightarrow{\rho} H$ induces a functor
$H\text{Set} \to G\text{Set}$
$H \circlearrowright X \qquad G \circlearrowright X$

(5) (i) Fix R a ring (possibly non-comm.) Let RMod be the cat of left Rmods

(ii) For any ring map $\rho: R \to S$ have functor SMod $\to$ RMod

(6) A unital monoid is the same thing as a category w/ one object.

(7) Assume all $\hom(X, Y)$ are abelian groups, and that composition is bilinear. Then $\hom(X, Y)$ is a $\hom(X, X) - \hom(Y, Y)$ bimodule $\forall X, Y$.

(8) Look up "Natural Transformation." Show {Nat. Transformation of the Identity Function of RMod} $\cong$ Center of R.

2/2/18. <u>Categories & Functions</u>

(I) Natural Transformations

(II) Equivalences of Categories.

	<u>obj</u>	<u>hom</u>
Gps.	gp G	$G \to G'$ gp homom
Cats	cat $\mathcal{C}$	$\mathcal{C} \to \mathcal{D}$ functor.

Fix $\mathcal{C}, \mathcal{D}$, study $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ fun $\mathcal{C} \xrightarrow{F} \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{D} \end{tikzcd}$$

natural transformation.

<u>Defn</u> Fix two functors $F, G : \mathcal{C} \to \mathcal{D}$. A <u>natural transformation</u> from F to G is the data of:

$\forall\ X \in \mathrm{ob}\,\mathcal{C}$ $\eta_X : F(X) \to G(X)$

such that $\forall\ f : X \to Y$ in $\mathcal{C}$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

this map commutes, i.e. $\eta_Y \circ F(f) = G(f) \circ \eta_X$

<u>Rmk/Exer</u>: $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ forms a category.

<u>Obs</u>: Set $\mathcal{C} = \mathcal{D}$. $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ has two interesting ways compose.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \mathcal{C} \arrow[r, "L"'] & \mathcal{C} \end{tikzcd}$$

horizontal & vertical compositions.

$$\rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "L \circ F"] \arrow[r, "L \circ G" description] \arrow[r, bend right=50, "L \circ H"'] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \mathcal{C} \end{tikzcd}$$

? When do we consider two categories the same?

Fix $X, Y \in \operatorname{ob}\mathcal{C}$.

<u>Defn</u> A morphism $f : X \to Y$ is called an <u>isomorphism</u> if $\exists\, g$ such that $X \xrightarrow{f} Y \xrightarrow{g} X$, $g \circ f = id_X$

and $Y \xrightarrow{g} X \xrightarrow{f} Y$, $f \circ g = id_Y$

ex. $id_X : X \to X$

Fix $\mathcal{C}, \mathcal{D}$.

<u>Defn</u> A functor $F : \mathcal{C} \to \mathcal{D}$ is called an <u>equivalence</u> of categories if

technical term

1) F is essentially surjective, i.e. $\forall\, Y \in \operatorname{ob}\mathcal{D}$, $\exists\, X \in \operatorname{ob}\mathcal{C}$ st $F(X) \cong Y$.

2) $\forall\, X_0, X_1 \in \operatorname{ob}\mathcal{C}$, $\hom(X_0, X_1) \to \hom(F(X_0), F(X_1))$ is a bijection.

$$\begin{array}{cc} X_0 & F(X_0) \\ \downarrow\downarrow\downarrow\downarrow\downarrow & \downarrow\downarrow\downarrow\downarrow\downarrow \\ X_1 & F(X_1) \\ \mathcal{C} & \mathcal{D}. \end{array}$$

☆ You can also ask for an isomorphism of categories, and it will be the same defn as above, but it's rarely used b/c it's too rigid.

<u>Exer</u> (Use Axiom of Choice) If $F : \mathcal{C} \to \mathcal{D}$ is an equiv of cats, then $\exists\, G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms (natural transf. st η_X is $\cong$) η_1, η_2 st

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G \circ F"] \arrow[r, bend right, "id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\eta_1"] & \mathcal{C} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "F \circ G"] \arrow[r, bend right, "id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\eta_2"] & \mathcal{D} \end{tikzcd}$$