EXERCISES IN CATEGORIES

Warning 0.0.1. Often, people define a category only by its objects, or a functor only by its objects. (Example: Let *Grp* be the category of groups, and let $Grp \to Grp$ be the functor sending any group to its abelianization.) This, strictly speaking, is *not* enough to determine a category or a functor. It is implicitly understood that there is a clear/natural/do-able definition of what the morphisms of the category are, or what the functor ought to do on morphisms.

Be warned that many first-timers make the mistake of defining a functor only on objects and realizing that there's no way to actually construct a functor (i.e., to define how their assignment is compatible with morphisms).

1. Algebraic notions encoded in categories

Suppose that $\mathcal{C}$ is a category such that, for every pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$, the set $\hom(X, Y)$ has been endowed with the structure of an abelian group. Moreover, suppose that the composition map $\hom(X, Y) \times \hom(Y, Z) \to \hom(X, Z)$ is $\mathbb{Z}$-linear in each variable. (The technical term is that $\mathcal{C}$ is *enriched* in abelian groups.)

1.1. Show that for any $X \in \operatorname{Ob}\mathcal{C}$, $\hom(X, X)$ is a (not necessarily commutative) unital ring.

1.2. Show that for any pair $X, Y \in \operatorname{Ob}\mathcal{C}$, $\hom(X, Y)$ is a bimodule over $\hom(X, X)$ and $\hom(Y, Y)$. More specifically, $\hom(X, Y)$ is a left module over $\hom(X, X)$ and a right module over $\hom(Y, Y)$, and these module actions commute.

2. Merging combinatorial definitions with important algebraic notions

2.1. **Commutative diagrams via linear posets.** Let $[k] = \{0 < 1 < \ldots < k\}$ be the linear poset with $k+1$ elements. Recall you can consider $[k]$ itself a category with $k+1$ objects, with $\hom(i, j) = *$ when $i \leq j$, and $\hom(i, j) = \emptyset$ otherwise. Show that for any category $\mathcal{D}$, the following are the same thing:

(1) a functor $[k] \to \mathcal{D}$
(2) a choice of commutative diagram in $\mathcal{D}$ in the shape of a k-simplex.

(3) a choice of sequence of objects X_i, $i = 0, \ldots, k$, and of composable morphisms, $f_{01}, \ldots, f_{(k-1)k}$, where $f_{ij} : X_i \to X_j$.

Remark 2.1.1. Sometimes, a combinatorist or a topologist may refer to $[k]$ as the *combinatorial k-simplex*, or sometimes even a k-simplex.

2.2. **Category of simplices and associative algebras.** Here, simplices show up in a *very different* way than above. Rings also show up in a *very different* way from the previous problems. Do not rely on the previous problems to do this one.

Let Δ_s be the category where an object is a finite, non-empty, linearly ordered poset, and where a morphism is a map of posets which is also a surjection. (You can check that any object is a poset isomorphic to $[k]$ for some $k \geq 0$. Also, recall that a map of posets is a function $f : P \to Q$ such that $p \leq p' \implies f(p) \leq f(p')$.

Let R be an associative ring. Show that R determines a functor

$$F_R : \Delta_s \to Ab$$

where Ab is the category of abelian groups, sending $[0]$ to R.

(Hint: It may help to just think about the subcategory of Δ_s only consisting of objects of the form $[0], [1]$, and $[2]$.)

3. Natural transformations

Fix $\mathcal{C}, \mathcal{D}$ two categories, and two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$.

Definition. A natural transformation η from F to G is a choice of homomorphism $\eta_X : F(X) \to G(X)$ for every $X \in \mathrm{Ob}\,\mathcal{C}$. This data must satisfy the following property: For any $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

commutes.

3.1. **You can compose natural transformations.** Fix three functors F, G, H from $\mathcal{C}$ to $\mathcal{D}$. Convince yourself that if η is a natural transformation from F to G, and if η' is a natural transformation from G to H, then there is a composition $\eta' \circ \eta$ which defines a natural transformation from F to H.

3.2. **For a fixed functor, natural transformations to/from itself are a monoid.** If $F = G = H$, convince yourself that the set of natural transformations forms an associative monoid with unit.

3.3. **For Sets.** Let $\mathcal{C}$ be the category of sets, and let $F = G$ be the identity functor from $\mathcal{C}$ to $\mathcal{C}$. Compute the set of all natural transformations of the identity (i.e., from F to G).

3.4. **For comm. rings.** Do the same, when $\mathcal{C}$ is the category of commutative unital rings.

4. Categories as invariants of objects you care about

One use of algebra is to translate a hard object into an algebraic object (i.e., construct an invariant which is workable).

Here, we will see a baby version: How to take a group G and create two different categories from it. We'll see that each category has some information about G.

4.1. **Center of G.** Let G be a group. Let $\mathcal{C} = GSets$ be the category of left G-sets; that is, an object is a set X with left G-action. A morphism is a map of G-sets, so a function $f : X \to X'$ such that $gf(x) = f(gx)$ for all $g \in G, x \in X$.

Show that the set of natural transformations of the identity functor can be identified with the *center* of G. Make sure you check that this identification respects composition (of natural transformations) and multiplication (of elements of G).

4.2. **Center of $k[G]$.** (This is more involved.) Fix a field k. Let G be a group. Let $\mathcal{C} = GMod$ be the category of left G-modules over k. That is, an object is a k-vector space with a k-linear G action—put another way, an object is a k-vector space V equipped with a group homomorphism $G \to Aut_k(V)$. Morphisms are k-linear maps $V \to V'$ respecting the group action.

Show that the set of natural transformations of the identity functor can be identified with the set of *class functions of* G. As a vector space, this is a vector space generated by the set of conjugacy classes of G. It may help to identify the class functions as the center of the group ring $k[G]$.

Remark 4.2.1. Note that "natural transformation" only knows about the category $\mathcal{C}$, and does not know that the category a priori came from a group G. So a purely categorical invariant (natural transformation of the identity) recovers something about the group (center, or class functions).

5. Morita invariance

This section shows that the invariants are not always so strong, but define an interesting equivalence relation on rings. First, let's define the notion of an equivalence of categories:

Definition 5.0.1. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. We say that f is an *isomorphism* if there exists $g : Y \to X$ for which $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. We say two objects are *isomorphic* if they admit an isomorphism between them.

Definition 5.0.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* of categories if:

(1) For every $X, Y \in \mathrm{Ob}\,\mathcal{C}$, the function $F : \hom(X, Y) \to \hom(F(X), F(Y))$ is a bijection.
(2) For every object $Z \in \mathrm{Ob}\,\mathcal{D}$, Z is isomorphic to some object of the form $F(X)$.

Here, we take a ring R, and produce a category as an invariant.

Definition 5.0.3. Fix a field k, and fix a unital (possibly non-commutative) k-algebra R. (This is a fancy terminology for a ring R equipped with a ring homomorphism $k \to R$.) We let $RMod$ denote the category of left R-modules which are k-linear. That is, an object is a k-vector space equipped with an R-module structure, and a morphism is a k-linear map respecting the R action.

5.1. **Matrix rings are Morita equivalent.** Let m, n be two integers ≥ 1. Let R_m be the ring of $m \times m$ matrices with entries in k. Likewise for R_n.

Show that the categories $R_m Mod$ and $R_n Mod$ are equivalent.

Remark 5.1.1. Two rings R, S such that $RMod$ and $SMod$ are equivalent are called *Morita equivalent.*

5.2. **Centers of rings are Morita invariant.** Show that if two rings do not have isomorphic centers, then they are not Morita equivalent. (Hint: Natural transformations of the identity.)

6. Some constructions

6.1. **Product posets.** Let P and Q be posets. How would you define a poset structure on $P \times Q$? Define one that's "functorial," in the sense that if $P \to P'$ and $Q \to Q'$ are maps of posets, then your construction induces a map of posets $P \times Q \to P' \times Q'$.

6.2. **Product categories.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Convince yourself that there's a category called $\mathcal{C} \times \mathcal{D}$, where

(1) $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$, and
(2) For $X, X' \in \mathrm{Ob}\,\mathcal{C}$ and $Y, Y' \in \mathrm{Ob}\,\mathcal{D}$, we have $\hom((X, Y), (X', Y')) = \hom(X, X') \times \hom(Y, Y')$.

6.3. **Product posets again.** Let *Poset* be the category of posets—an object is a poset, and a morphism is a map of posets (a function $f : P \to P'$ such that $p \leq q \implies f(p) \leq f(q)$). Show that your construction of product posets defines a functor from $Poset \times Poset$ to $Poset$.

6.4. **1 times 1.** Fix a category $\mathcal{D}$, and let $[1] = \{0 < 1\}$ be the usual poset. What does a functor $[1] \times [1] \to \mathcal{D}$ encode?

6.5. **Opposites.** Let $\mathcal{C}$ be a category. Then define a new category $\mathcal{C}^{\text{op}}$ by the following:

(1) $\text{Ob}\,\mathcal{C}^{\text{op}} = \text{Ob}\,\mathcal{C}$.
(2) $\hom_{\mathcal{C}^{\text{op}}}(X, Y) = \hom_{\mathcal{C}}(Y, X)$. (Here, the subscript indicates in which category we are considering morphisms. In English: The set of morphisms from X to Y in $\mathcal{C}^{\text{op}}$ is the set of morphisms from X to Y in $\mathcal{C}$.)
(3) I leave to you to define the composition.

Show that $\mathcal{C}^{\text{op}}$ is indeed a category.

6.6. **Contravariant functors.** A "contravariant funtor from $\mathcal{C}$ to $\mathcal{D}$" is a functor $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$.

Convince yourself that the assignment $G \mapsto GSets$ is a contravariant functor from *Grp* to *Cat*—i.e., from the category of groups (and group homomorphisms) to the category of categories (and functors).

6.7. **Categories of functors.** Fix $\mathcal{C}, \mathcal{D}$ two categories, and let $Fun(\mathcal{C}, \mathcal{D})$ be the following category:

(1) An object is a functor $F : \mathcal{C} \to \mathcal{D}$,
(2) A morphism from F to F' is a natural transformation.

Convince yourself that $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a category.

7. Some tests

7.1. **Units.** Let *Ring* be the category of (possibly non-commutative) unital rings, and *Grp* the category of groups. Consider the assignment which sends any unital ring R to its group of units $R^\times$. Can this be made into a functor $Ring \to Grp$? How about $Ring^{\text{op}} \to Grp$?

7.2. **Idempotents.** Let $\mathcal{C}$ be the category with one object, which has exactly two morphisms: id and f, with composition defined by $f^2 = f$. For a fixed field k, construct/define a category whose objects are pairs (V, V_0) where V is a finite-dimensional k-vector space and V_0 is a subspace. (What are the morphisms?) Exhibit an equivalence between your category, and the functor category $Fun(\mathcal{C}, Vect_k^{fd})$ where $Vect_k^{fd}$ is the category of finite-dimensional k vector spaces.

7.3. **Abelianize.** Let $Ab : Grp \to Grp$ be the functor that sends any group to its abelianization. (You should check that this is a functor—i.e., how is it defined on morphisms?) Can you compute the monoid of natural transformations from Ab to itself?

7.4. **Equivalences.** Show that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if there exists a functor $G : \mathcal{D} \to \mathcal{C}$, together with natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$. (A natural isomorphism is a natural transformation where every η_X is an isomorphism.)