Syllabus for Reading Seminar: Toward Fukaya Categories

0.1. Organization and timeline. This syllabus is scheduled for a semester of about 14 weeks (which includes vacation time as one of those weeks). Roughly 2 weeks will be spent on each section, hence the 7 sections.

It's my hope that we will go a bit fast toward the beginning; for instance, finishing topics 1-4 within seven weeks (rather than eight) would give us more time to discuss other ideas toward the end.

0.2. Format. Except for the fact that I have chosen the below outline of topics, the seminar will be heavily student-driven. This means you will get out what you put in.

Also, every talk will be given by a student. Every participant will sign up at the beginning of the semester to take responsibility for teaching everybody some topic N. Ideally, there will be two to three students taking responsibility for each topic. Necessarily, these teachers will give talks about the topics to introduce the topic to the class. A speaker for topic N (which will roughly begin at week $2N$) is strongly encouraged to start learning and preparing for the topic by week $2N-2$, and to even give a "practice talk" to peers before giving the talk in the seminar. Ideally, every week will be structured as follows:

Monday: Student talk introduces some part of a topic. Gives basic definitions and examples. Student talks will most likely have interruptions by meaningful questions.

Wednesday: Continue student talks as necessary, then delve into discussion and examples—perhaps driven by random exercises in textbooks, or by proofs of statements students didn't understand, or just by trying to tinker. For instance, if a definition was introduced, can you construct examples as a team? If examples were introduced, can you understand them and ask questions of them?

Friday: Continue exploring topics as necessary. If not necessary, the next student talk should be given (so you should be ready to give your talk one session early, just in case). Repeat.

0.3. Mathematical Content. This reading seminar's mathematical goal is to build up to a topic called the homological mirror symmetry conjecture—a conjecture stated by Maxim Kontsevich in 1994. (You can Google him or his conjecture if you like.) This conjecture comes from observations in theoretical physics. Roughly, it claims that two very different kinds of algebraic invariants—one kind coming from "symplectic geometry," and another kind coming from "algebraic geometry," are in fact equivalent. Some experts expect the conjecture to be proven in the next few decades, and indeed, some powerful techniques are being developed at present.

This seminar will only attempt to *define* the "symplectic geometry" side of this conjecture. (Even then, we will not fully define it.) Along the way, we'll be exposed to ideas, tools, and philosophies that are fairly prevalent in modern mathematics.

Sections 1-4 are basic topics that most mathematicians with a Ph.D. will know, or have seen at one point in their lives.

Section 5 is a bit more specialized, but is a very natural language for encoding classical mechanics. So it may be useful for physics majors.

Sections 6-7 are very specialized, and are specific to possible summer topics for research/reading.

0.4. Skill-set Goals. Here are some skills I hope you will gain from this seminar. Many of these are pre-professional, in that they would serve you very well should you go onto graduate school in any field. More broadly, these could be life skills for delving into any field of knowledge.

- Public speaking.
- Learning without always being given proper context. I will try to give context as appropriate, but having to figure *some* of this out on your own is a big part of the process of independent learning.
- Collaborating and learning with others.
- Asking questions.
- Knowing what you do and do not understand, and bridging the gap as necessary.

0.5. Guideline for speakers. Giving good math talks is a skill, and a very difficult one. Most mathematicians need to learn the basic mechanics of public speaking (good boardwork, volume, speed of speech), and also think about how to best structure a talk (how to introduce new ideas, when to give examples, how to state a definition).

For this reason, I would suggest giving a practice talk to a few math friends and other participants before giving a talk in the seminar—you should give these practice talks to math friends who are willing to tell you things like: "I think you should change this part."

Also, for this seminar, you should provide everybody else in the seminar with some references. Where did you learn your topic? This way, even if some aspect of your talk is not clear, some students can still look up the material and try to learn. (Sounds similar to taking a class, no?) Every speaker/team should also provide notes for the topics they teach us; these will be posted online.

0.6. Being good at looking things up. For most topics here, a good Google search will yield fine resources, especially PDFs of course notes. However, every now and then you will find that the best resources are books/published articles. It is natural to initially feel that most books/articles are inaccessible (and that feeling will not go away; researchers always have to learn a lot to understand the works of peers). But a good resource is Mathscinet. If you are logged in via Harvard, Mathscinet gives you access to reviews of published works. These reviews are written by peer mathematicians, and often give a summary of the context of the work, and the implications thereof, without too much technical detail.

0.7. Course Credit. My understanding is that you will register for a 91r course in the math department. (Historically, 91r is a one-on-one reading seminar, but I have been recommended to pursue this course number.) You will get a formal letter grade, based on your participation, anything you write, and talk(s) in the class. This seminar does not count toward concentration credit.

1. Chain complexes and cochain complexes (One week)

Chain complexes are a basic algebraic tool—now-a-days they are as basic as abelian groups. They are extremely useful: For example, to every space one can assign a chain complex, and hence invariants of this chain complex become invariants of the space.

A chain complex is the data of an abelian group A_i for every $i \in \mathbb{Z}$, and an abelian group map $d_i : A_i \to A_{i-1}$ for every i, such that $d^2 = 0$. (That is, $d_i \circ d_{i+1} = 0$ for every i.)

The *ith homology group* of a chain complex is the quotient group

$$kernel(d_i)/image(d_{i+1}).$$

You should tell us what a chain map (aka a map of chain complexes) is, and what a homotopy of chain maps is. You should prove that two maps that are chain-homotopic induce the *same* map on homology groups. Define quasi-isomorphism of chain complexes—this is *like* an isomorphism of chain complexes.

Further, given two chain complexes A and B, you should define the morphism chain complex, denoted $hom(A, B)$. This is itself a chain complex.

A chain complex is more or less the same thing as a cochain complex, just flipped upside down. You should explain this.

Talk about how it makes sense to talk about chain complexes of R-modules for any commutative ring R. (This can come at the very beginning, in fact.)

Give simple examples of chain complexes. Give examples of quasi-isomorphisms that do not admit inverse quasi-isomorphisms. However, prove that if a chain complex is made up of *free* R-modules (so that each A_i is free) then show that every quasi-isomorphism admits an inverse (possibly up to chain homotopy, depending on your construction).

1.1. Homology groups and quasi-isomorphisms.

1.2. Chain homotopy.

1.3. Hom complex.

1.4. Make some examples by hand.

References.

- Weibel, Homological algebra.
- Hatcher, Algebraic topology.

2. Smooth Manifolds (One or two weeks)

Define what open balls are in $\mathbb{R}^n$. An open subset U of $\mathbb{R}^n$ is a union of open balls in $\mathbb{R}^n$—possibly an infinite union, and possibly an empty union. A closed subset of $\mathbb{R}^n$ is a complement of an open set.

Fix a subset $X \subset \mathbb{R}^n$. Then an *open set* of X is any subset of X obtained as an intersection $X \cap U$ where $U \subset \mathbb{R}^n$ is open.

Define what a topological space is. Under the above two definitions, show that both $\mathbb{R}^n$ and X are topological spaces.

Define what a topological manifold is.

Define what smooth functions from $\mathbb{R}^n \to \mathbb{R}$ are, and hence smooth functions from any open subset of $\mathbb{R}^n$ to $\mathbb{R}$. Then define a smooth function from $\mathbb{R}^n \to \mathbb{R}^m$. Show that smooth functions are closed under composition.

Define what a *smooth atlas* on a topological manifold is. Define smooth manifold. Show that this allows you to define what a smooth function on a smooth manifold is. The notion of "isomorphism" for smooth manifolds is diffeomorphism. Define diffeomorphism.

State but don't prove Whitney's embedding theorem; so you can consider any smooth manifold as a submanifold of some $\mathbb{R}^n$. Use the implicit function theorem to show how certain subsets of $\mathbb{R}^n$ naturally become smooth submanifolds. Assuming the implicit function theorem or the submersion theorem, give a one-line proof that $S^n \subset \mathbb{R}^{n+1}$ is a smooth manifold. Explain how derivatives of smooth maps induce linear maps between tangent spaces. (Tangent spaces will be easier to define if you assume your manifolds are embedded in $\mathbb{R}^n$.)

However, some smooth manifolds do not arise easily as submanifolds of $\mathbb{R}^n$. Give the example of Grassmanians as manifolds.

2.1. Smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

2.2. Topological spaces given by subsets of $\mathbb{R}^n$ for some n.

2.3. Open sets and closed sets; definition of topological space.

2.4. Charts and atlases, why do we need them?

2.5. Smooth atlases (definition of a smooth manifold).

2.6. Smoothness of functions on a manifold.

References.

- Guillemin and Pollack, Differential Topology.
- Milnor, Morse Theory.
- Warner, Foundations of Differentiable Manifolds and Lie Groups.

3. Morse Theory (One or two weeks)

Define what a (smooth) Riemannian metric on a smooth manifold is. Given a Riemannian metric and a smooth function f on a smooth manifold, define the gradient vector field. Define what a critical point of f is, and define what a gradient trajectory is.

Define what a Morse function is.

Define what it means for a subset of $\mathbb{R}^n$ to be compact. (There is another abstract definition of compactness you can give if you like, but the intuition of being closed and bounded is enough.)

Define the Morse chain complex. Make sure to talk about how a single gradient trajectory gives rise to infinitely many if the trajectory is not constant.

Define what the singular homology/singular chain complex of a topological space is; state (and sketch a proof if you know how) the result that the homology of the Morse complex is isomorphic to the homology of the manifold when the manifold is compact.

The most difficult part of this talk is understanding why the Morse chain complex is indeed a chain complex (i.e., why $d^2 = 0$). It's also a fun proof.

3.1. Riemannian metrics.

3.2. Gradient vector fields. Examples.

3.3. Compactness.

3.4. The set of gradient vector fields, modulo $\mathbb{R}$.

3.5. Morse chain complex.

3.6. Singular chains of a space.

References.

- Milnor, Morse Theory.
- Schwarz, Morse homology.
- Bott, Morse theory indomitable.
- Schwarz, Equivalences for Morse homology.
- Bott, Lectures on Morse theory, old and new.

4. Differential forms (One or two weeks)

Define smooth 1-forms on a smooth manifold. Define tensor product—as a word of caution, tensor products are almost always understood using their universal properties. Then define exterior tensor product, or wedge product, again paying attention to universal properties. Define smooth k-forms; definitely write out what these look like in local coordinates. Define the deRham derivative, otherwise known as the exterior differential.

Define cdga (commutative differential graded algebra) and show that the deRham cochain complex of differential forms is a cdga. Its cohomology is called the deRham cohomology of a smooth manifold.

You can give some intuition by saying that a k-form gives a way of measuring volumes of infinitesimal k-dimensional parallelipipeds on a manifold, and that a top-form gives a way of measuring volumes.

For us, the most important kinds of forms will be 0, 1, 2, and top-forms. Explain how to take derivatives of such forms carefully.

Try to give an example that shows that an equation like $d\alpha = 0$ for a 4-form can encode a very sophisticated system of differential equations. The most famous examples are from special relativity. Another way in which d is a useful operator is in understand div, grad, and curl. Explain how d gives rise to these operations in $\mathbb{R}^3$, and what $d^2 = 0$ means in this setting. Note that although d is an operation on *forms*, you obtain div/grad/curl (which deal with vector fields) because a Riemannian metric on $\mathbb{R}^3$ allows you to identify vectors with forms.

4.1. Tensor products of vector spaces; universal properties.

4.2. Exterior algebra. Antisymmetry is graded symmetry. Universal properties.

4.3. Dual vector spaces over $\mathbb{R}$.

4.4. Vector fields on a smooth manifold.

4.5. 1-forms on a smooth manifold.

4.6. k-forms on a smooth manifold.

4.7. deRham derivative (exterior derivative).

4.8. deRham algebra is a cdga.

References.

- Guillemin and Pollack, Differential Topology.
- Warner, Foundations of Differentiable Manifolds and Lie Groups.

5. Symplectic geometry (Two weeks)

Define a symplectic form on a vector space.

Define symplectic manifold. A symplectic manifold M is a setting in which one can do classical geometry.

Define cotangent bundle of a smooth manifold X. Show it is an example of a symplectic manifold M.

Define what the Hamiltonian vector field associated to a function $H : M \to \mathbb{R}$ is. Show how the harmonic oscillator is associated to a function $x^2 + y^2 = 1$, writing $T^*\mathbb{R} \cong \mathbb{R}^2$.

Define Lagrangian submanifold $L \subset M$. Show that flowing by a Hamiltonian vector field sends Lagrangians to Lagrangians.

State (and prove if you can) Weinstein and Darboux-Weinstein theorems. This tells you what symplectic manifolds and their Lagrangians locally look like.

One of the ideas here is that symplectic manifolds are hard to study because they're all locally the same; also, Lagrangians are some of the best tools we have to study symplectic manifolds, but it's hard to construct them. You should definitely tinker to try and make some.

Good examples are $\mathbb{C}P^n$ for every $n \geq 0$. $\mathbb{C}P^2$ is already quite interesting, so just define this example and its symplectic form, give at least two examples of Lagrangians in $\mathbb{C}P^2$.

5.1. Cotangent bundles.

5.2. Hamiltonian formulation of physics, examples.

5.3. Definition of symplectic forms and Lagrangians. Examples of Lagrangians.

5.4. Hamiltonian isotopies.

5.5. $\mathbb{C}P^n$.

References.

- D. McDuff, D. Salamon. Introduction to symplectic topology. Claredon, 1995
- Dusa McDuff: What is symplectic geometry?, 2009. `www.math.sunysb.edu/~dusa/ewmcambrevjn23.pdf`
- V. Arnold. Symplectic geometry and topology. J. math. Phys., 41 (2000), 3307-3343
- V. Arnold, A. Givental. Symplectic geometry. Dynamical System 4, Springer
- R. Berndt. An introduction to symplectic geometry. AMS, 2001

- Ya. Eliashberg, L. Traynor, ed. Symplectic geometry and topology. AMS, 1999
- H. Hofer, E. Zehnder. Symplectic invariants and Hamiltonian dynamics. Birkhauser, 1994

6. Holomorphic disks and Floer Theory (Two weeks)

Define what an almost-complex structure is. Define what it means for an almost-complex structure on a symplectic manifold M to be *compatible* (with the symplectic structure). Define what a holomorphic strip $\mathbb{R} \times [0,1] \to M$ is with boundary on two Lagrangians $L_0 \subset L_1$.

Define the Floer cochain complex. You can try to ignore issues of grading if you want, or you can jump right in. Explain roughly why the Floer cochain complex computes Morse homology. How do you know $d^2 = 0$?

Give some historical context as to why holomorphic curves and Lagrangians became so important in symplectic geometry. The main ideas here are due to Gromov—his proof of non-squeezing, and the non-existence of certain kinds of Lagrangians in $\mathbb{R}^{2n}$. His papers are notoriously difficult to read, so I would suggest trying to find other sources which comment on/review his works.

6.1. Almost-complex structures.

6.2. Holomorphic curve equation.

6.3. Floer cochain complex.

6.4. Connection to Morse theory for cotangent bundles.

References.

- Floer, Morse theory for Lagrangian intersections
- And: `http://www.comp.tmu.ac.jp/pseudoholomorphic/akaho_a_crash_course_of_floer_homology_for_lagrangian_intersections.pdf`
- M. Gromov, Pseudo holomorphic curves in symplectic manifolds. Inventiones Mathematicae vol. 82, 1985, pgs. 307-347.
- Donaldson, Simon K. (October 2005). ”What Is...a Pseudoholomorphic Curve?”. Notices of the American Mathematical Society. 52 (9): pp.1026—1027.
- See also: `http://math.berkeley.edu/~auroux/290f11/C.%20Gerig%20-%2022-09-11%20-%20Lagrangian%20Floer%20homology.pdf`
- Notes from this class: `http://math.mit.edu/~auroux/18.969-S09/`

7. Fukaya categories (One or two weeks depending on available time)

Define categories and A_∞-categories. Sketch the definition of the Fukaya category of a symplectic manifold. State the homological mirror symmetry conjecture.

The general feel is that the Fukaya category, like most categories, is fairly intractable, in the following sense: We rarely understand a big algebra/ring by writing down all its elements and explicitly understanding all the multiplications. We often try to find tractable generators, or develop some feel for the "place" of the ring among other rings by understanding certain homomorphisms or universal properties.

Likewise, techniques like "finding generators" or "understanding relations with other categories" is what allows Fukaya categories to be understandable; even though the *geometry* captures a giant collection of Lagrangians and a bunch of holomorphic disk data, and we will certainly *never* write out every Lagrangian or every holomorphic disk, once the category is defined, one can try to understand the *algebraic* place of this category in the math world.

7.1. Categories.

7.2. Categories enriched in chain complexes.

7.3. Holomorphic disks with many marked points on boundary.

7.4. A_∞-categories.

7.5. Statement of mirror symmetry conjecture.

References.

- Notes from this class: `http://math.mit.edu/~auroux/18.969-S09/`