Geodesics as locally minimizing

Prop Let (M, g) be Riemannian. Then $\forall\, p \in M$, $\exists$ open set $\mathcal{U} \subset M$, $p \in \mathcal{U}$, for which:

If $\gamma : [0,1] \to \mathcal{U}$ is a geodesic w/ $\gamma(0) = p$, $\gamma(1) = q$

and $c : [0,1] \to M$ is a piecewise linear curve w/ the same endpts,

then
$$\text{length}(\gamma) \leq \text{length}(c),$$
and
$$\text{length}(\gamma) = \text{length}(c) \Rightarrow \text{image}(\gamma) = \text{image}(c).$$

So locally, geodesics minimize length.

Pf You'll prove this on your final. //

Rmk Often, $\mathcal{U} \neq M$. For instance, if $M = S^2$, take γ to be a great circle of length $> \pi R$. Then γ does not minimize length.

(M, g) as a metric space.

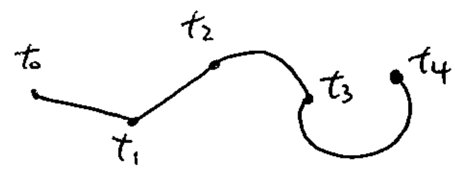
Defn Let M be a manifold. A piecewise smooth curve is a continuous map

$$\gamma : [a,b] \longrightarrow M$$

s.t. $\exists$ finitely many t_i, $a = t_0 < t_1 < \cdots < t_n = b$ for which

$$\gamma : [t_i, t_{i+1}] \longrightarrow M$$

is smooth.



The length of a piecewise smooth curve γ is the number

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\dot{\gamma}| \, dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \, dt.$$

If (M, g) is Riemannian, let

$$d: M \times M \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

$$(p, q) \longmapsto \inf_{\substack{\text{piecewise-smooth} \\ \gamma \text{ s.t. } \gamma(a) = p \\ \gamma(b) = q}} \text{length}(\gamma).$$

<u>Prop</u> If M is connected, d makes M into a metric space.

<u>Pf</u> Triangle inequality and symmetry are easy:

Δ ineq: $$d(p, r) = \inf_{\gamma} \text{length}(\gamma) \leq \inf_{\substack{\gamma \text{ that} \\ \text{pass through } q}} \text{length}(\gamma)$$

$$= \inf_{\substack{\gamma \text{ passing through} \\ q}} \text{length}(\gamma_1) + \text{length}(\gamma_2)$$

$$\leq \inf_{\substack{\gamma_1 \text{ from } p \\ \text{to } q}} \text{length}(\gamma_1) + \inf_{\substack{\gamma_2 \text{ from} \\ q \text{ to } r}} \text{length}(\gamma_2)$$

$$= d(p, q) + d(q, r).$$

Symm: A path from p to q determines a path from q to p, of equal length.

Obviously, $d(p,q) \geq 0$ $\forall$ $p,q \in M$, since length ≥ 0 for all curves.

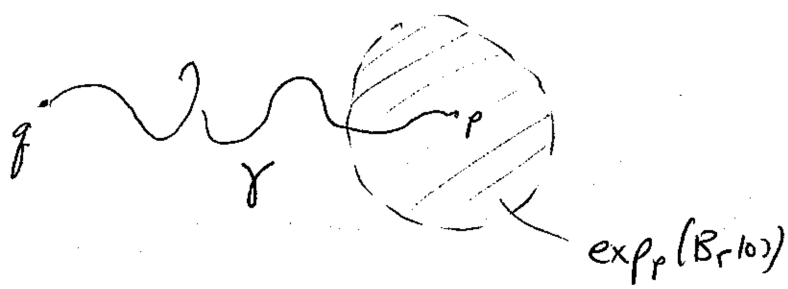
Now let's prove

$$d(p,q) = 0 \implies p = q. \quad \text{(We'll prove the contrapositive.)}$$

If $p \neq q$, fix $r > 0$ so that

- $\exp_p(B_r(0))$ is an open set for which geodesics from p minimize length, and
- $q \notin \exp_p(B_r(0))$.

Then $\forall$ piecewise linear $\gamma : [a,b] \to M$, w/ $\gamma(a) = p$, $\gamma(b) = q$,



$$\gamma^{-1}(\exp_p(B_r(0)))$$

is a segment of γ that must have length $\geq r$. Hence

$$\inf_\gamma \text{length}(\gamma) \geq r > 0. \quad //$$

Hopf-Rinow Thm

Let M be a connected Riemannian manifold. If $\exists\, p \in M$ s.t. $\exp_p$ is defined on all of T_pM, then $\forall$ ~~[illegible]~~ $q \in M$, $\exists$ geodesic connecting p to q. Moreover, ~~[illegible]~~ one can find a geodesic γ such that

$$\text{length}(\gamma) = d(p,q).$$

Pf Final. //

Rmk This doesn't mean all geodesics are globally length-minimizing; rather, one can always find some geodesic which is minimizing (for complete manifolds).

Cor Suppose (M, g) is connected and $\exists\, p \in M$ for which $\exp_p$ is defined on all of T_pM. Then any closed, bounded subset of M is compact.

Pf: $L \subset M$ bounded $\Longrightarrow$ $\exists\, d \in \mathbb{R}$ s.t. $d(p,x) \leq d \quad \forall\, p \in L$.

By Hopf-Rinow, the closed ball of radius d $\quad \overline{B_d(0)} \subset T_pM$

has

$$\exp_p\left(\overline{B_d(0)}\right) \supset L.$$

But $\exp_p$ is continuous and $\overline{B_d(0)} \subset T_pM \cong \mathbb{R}^{\dim M}$ is compact, so $\exp_p\left(\overline{B_d(0)}\right)$ is compact. If L is a closed subset of a compact subset, L itself is compact. //

Prop Let $(\widetilde{M}, \widetilde{g})$ be complete, and (M, g) Riemannian. If $p : \widetilde{M} \to M$ is a local diffeo s.t.

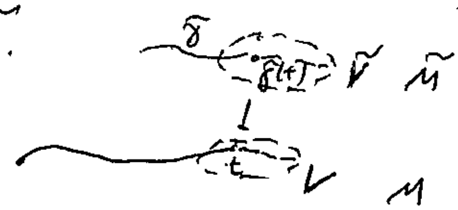
$$|T_p(\vec{v})| \geq |\vec{v}| \quad \forall\ \vec{v} \in T\widetilde{M}$$

then $\forall\ \gamma : [a,b] \to M$ continuous, and

$$\forall\ \widetilde{q} \in p^{-1}(\gamma(a))$$

$$\exists!\ \widetilde{\gamma} : [a,b] \to \widetilde{M} \quad \text{s.t.} \quad p \circ \widetilde{\gamma} = \gamma.$$

Pf. Since p is a local diffeo, if $\exists\ t \in [a,b]$ s.t. we have a lift $\widetilde{\gamma} : [a,t] \to \widetilde{M}$, then $\exists\ \varepsilon$ s.t. $\widetilde{\gamma}$ extends to a lift $[a, t+\varepsilon) \to \widetilde{M}$.



So the set of such t is open. Since $\widetilde{q} \in p^{-1}(\gamma(a))$, $a \in \{t\}$, so the set of such t is non-empty.

We show it's also closed: Note that if $T \in [a,b]$, then

$$\infty > \int_a^T |\dot{\gamma}(t)|\, dt = \int_a^T |T_p\, \dot{\widetilde{\gamma}}(t)|\, dt$$
$$\geq \int_a^T |\dot{\widetilde{\gamma}}(t)|\, dt$$

so if $t_n \to t_\infty \in [a,b]$, $\{\widetilde{\gamma}(t_n)\} \subset$ some closed, bounded set (e.g., $\exp_{\widetilde{\gamma}(a)}(\overline{B_{\mathrm{length}(\gamma)}(0)})$.)
$\Longrightarrow \widetilde{\gamma}(t_n) \to$ some $\widetilde{q} \in \widetilde{M}$. By continuity, $\widetilde{\gamma}(t_\infty) =: \widetilde{q}$ defines a lift of γ at t_∞. //

<u>Lemma</u> Let $p: \tilde{M} \to M$ be a continuous map where

- p is a surjective local homeo
- $\tilde{M}$ is locally path-connected
- M is locally simply connected
- p satisfies unique path-lifting property.

Then p is a covering map.

<u>Pf</u> ~~#~~ Fix $x \in M$. Let $V \subset M$, $x \in V$ be an open set which is simply connected.

Write
$$p^{-1}(V) = \coprod \tilde{V}_\alpha$$
where each $\tilde{V}_\alpha \subset \tilde{M}$ is a path-connected component of $p^{-1}(V)$. We will show
$$p|_{\tilde{V}_\alpha} : \tilde{V}_\alpha \longrightarrow V$$
is a homeo $\forall \alpha$.

- Surjection: ~~$\forall y \in V$, fix $\gamma: [0,1] \to V$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.~~ ~~By path-lifting~~ Since $\tilde{V}_\alpha \subset p^{-1}(V)$, $\exists$ $\tilde{y}_0 \in \tilde{V}_\alpha$, $y_0 \in V$ s.t. $p(\tilde{y}_0) = y_0$. $\forall$ $y_1 \in V$, fix a path $\gamma: [0,1] \to V$ s.t. $\gamma(0) = y_0$, $\gamma(1) = y_1$. By unique path-lifting, $\exists!$ $\tilde{\gamma}: [0,1] \to \tilde{V}_\alpha$. (Since $\tilde{V}_\alpha$ is path-connected, the lift $\tilde{\gamma}: [0,1] \to \tilde{M}$ factors through $\tilde{V}_\alpha$.) By def'n, $p \circ \tilde{\gamma}(1) = y_1$, so this shows surjection.
- Injection: OTOH, since V is simply-connected, any choice of γ, γ' are homotopic rel endpoints. So $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$, hence the map $y_1 \mapsto \tilde{\gamma}(1)$ is independent of choice of γ in V. This gives an inverse to p.
- Inverse continuous: Since p is a local homeom. //

<u>Thm</u> Let (M,g) be complete, and $K(x,y) \le 0$ $\forall$ $p \in M$, $x,y \in T_pM$.

Then $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map.

<u>Pf</u> We saw that $K(x,y) \le 0 \Longrightarrow$ ① $\exp_p$ is a local diffeo. (Hence local homeom.)

Since (M,g) is complete, Hopf-Rinow $\to$ ② $\exp_p$ is a surjection.

Now, since $\exp_p$ is a local diffeo, the metric on M pulls back to a metric on T_pM. This makes T_pM into a complete Riemannian mfld (final!).

Moreover, by definition of pull-back metric,

$$|T\exp_p(\vec{v})| = |\vec{v}| \quad \forall \vec{v} \in T(T_pM).$$

Hence we can use the proposition: ③ $\exp_p$ satisfies path-lifting property.

So we have ①,②,③ — since $\exp_p$ is a map between manifolds, this implies $\exp_p$ is a covering map. //