Lecture 35: Hadamaard's Theorem and Jacobi Fields

Last time I stated:

Theorem 35.1. Let (M, g) be a complete Riemannian manifold. Assume that for every $p \in M$ and for every $x, y \in T_pM$, we have

$$K(x, y) \leq 0.$$

Then for any $p \in M$, the exponential map

$$\exp_p : T_pM \to M$$

is a covering map.

1. Jacobi fields—surfaces from families of tangent vectors

Last time I also asked you to consider the following set-up: Let $w : (-\epsilon, \epsilon) \to T_pM$ be a family of tangent vectors. Then for every tangent vector $w(s)$, we can consider the geodesic at p with tangent vector $w(s)$. This defines a map

$$f : (-\epsilon, \epsilon) \times (a, b) \to M, \qquad (s, t) \mapsto \gamma_{w(s)}(t) = \exp_p(tw(s)).$$

Fixing $s = 0$, note that $f(0, t) = \gamma(t)$ is a geodesic, and there is a vector field

$$\left.\frac{\partial}{\partial s}\right|_{(0,t)} f$$

which is a section of γ^*TM—e.g., a smooth map

$$J : (a, b) \to \gamma^*TM.$$

Proposition 35.2. Let

$$J(t) = \left.\frac{\partial}{\partial s}\right|_{(0,t)} f.$$

Then J satisfies the differential equation

$$\nabla_{\partial_t}\nabla_{\partial_t} J = \Omega(\dot\gamma(t), J(t))\dot\gamma(t).$$

Chit-chat 35.3. As a consequence, the acceleration of J is determined completely by the curvature's affect on the velocity of the geodesic and the value of J.

Definition 35.4 (Jacobi fields)**.** Let γ be a geodesic. Then any vector field along γ satisfying the above differential equation is called a *Jacobi field*.

Remark 35.5. When confronted with an expression like

$$\nabla_X Y \qquad \text{or} \qquad \nabla_X \nabla_Y$$

there are only two ways to swap the order of the differentiation: Using the fact that one has a torsion-free connection, so

$$\nabla_X Y = \nabla_Y X + [X, Y]$$

or by using the definition of the curvature tensor:

$$\nabla_X \nabla_Y = \nabla_Y \nabla_X + \nabla_{[X,Y]} + \Omega(X, Y).$$

Why is this "swapping?" For instance, in the latter equation, we have replaced the operation of taking the covariant derivative in the Y direction, then in the X direction, by an expression involving covariant derivatives in the opposite order (along with other terms, of course).

Remark 35.6. Finally, since ∂_s, ∂_t are coordinate vector fields, we have

$$[\partial_s, \partial_t] = 0.$$

PROOF. First note that $f(0, t)$ is a geodesic, so we have

$$\nabla_{\partial_t} \frac{\partial f}{\partial t} = 0.$$

To be precise, the above equation is an equation taking place in the space of sections of $\gamma^* TM$—one could equivalently write it as

$$(\gamma^* \nabla)_{\frac{\partial}{\partial_t}} \dot{\gamma}(t) = 0.$$

Since $\nabla_{\partial_t} \frac{\partial f}{\partial t}$ is constant, we also have

$$\nabla_{\partial_s} \nabla_{\partial_t} \frac{\partial f}{\partial t} = 0.$$

Now we play the "swapping vector fields" game:

$$\begin{aligned}
0 = \nabla_{\partial_s}\nabla_{\partial_t}\frac{\partial f}{\partial t} &= \nabla_{\partial_t}\nabla_{\partial_s}\frac{\partial f}{\partial t} + \nabla_{[\partial_s,\partial_t]}\frac{\partial f}{\partial t} + \Omega(\partial_s,\partial_t)\frac{\partial f}{\partial t} \\
&= \nabla_{\partial_t}\nabla_{\partial_s}\frac{\partial f}{\partial t} + \nabla_0\frac{\partial f}{\partial t} + \Omega(\partial_s,\partial_t)\frac{\partial f}{\partial t} \\
&= \nabla_{\partial_t}\nabla_{\partial_s}\frac{\partial f}{\partial t} + \Omega(\partial_s,\partial_t)\frac{\partial f}{\partial t} \\
&= \nabla_{\partial_t}\nabla_{\partial_t}\frac{\partial f}{\partial s} + \nabla_{\partial_t}[\partial_t,\partial_s]f + \Omega(\partial_s,\partial_t)\frac{\partial f}{\partial t} \\
&= \nabla_{\partial_t}\nabla_{\partial_t}\frac{\partial f}{\partial s} + \Omega(\partial_s,\partial_t)\frac{\partial f}{\partial t}.
\end{aligned}$$

Using the fact that Ω is a 2-form (so swapping the order of the vector fields results in a sign change) we arrive at

$$\nabla_{\partial_t}\nabla_{\partial_t}\frac{\partial f}{\partial s} = \Omega(\partial_t,\partial_s)\frac{\partial f}{\partial t}.$$

This is an equation in the pullback bundle γ^*TM. Expressing the above as a differential equation taking place in TM, we have

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J(s) = \Omega(\dot\gamma, J(t))\dot\gamma.$$

□

2. Singularities of the exponential map

Now, to prove $\exp_p$ is a covering map, let's first prove it's a local diffeomorphism. Which is to say we need to prove that the derivative is nowhere singular.

Lemma 35.7. Let $w \in T_pM$ be a point at which $T\exp_p|_w$ has non-trivial kernel. If $w' \neq 0 \in T_w(T_pM)$ is in the kernel, then there exists a Jacobi field J along the geodesic γ_w such that

(1) $J(0) = J(1) = 0$
(2) J is non-zero somewhere.

Proof. Choose any path $w(s)$ inside T_pM such that $w(0) = w$ and $\frac{dw}{dt} = w'$. As before, consider the map

$$f(s,t) = \exp_p(tw(s)) = \gamma_{tw(s)}(1) = \gamma_{w(s)}(t).$$

Then we know that $J(t) := \frac{\partial f}{\partial s}$ is a Jacobi field. Note that at $t = 0$, $f(s,0)$ is a constant map, so its s-derivative is zero. Also note that at $t = 1$, we have that

$$\begin{aligned} J(1) &= \frac{\partial}{\partial s}\bigg|_{(0,1)} f \\ &= T\exp_p|_{f(0,1)} w' \\ &= T\exp_p|_w w' \\ &= 0. \end{aligned}$$

Moreover, J is not everywhere zero–$J(t)$ is given by

$$J(t) = T\exp_p|_{tw(0)}(tw')$$

while $\exp_p$ is a diffeomorphism near the origin—hence for t positive and small enough, the above must be non-zero since w' is non-zero. $\square$

Proposition 35.8. Assume that (M, g) is a Riemannian manifold whose sectional curvature always satisfies

$$K(x, y) \leq 0$$

for any $x, y \in T_pM$ for any $p \in M$. Then $\exp_p$ is a local diffeomorphism for any $P \in M$.

Proof. We claim that under the curvature hypothesis of this proposition, the function

$$||J(t)||^2 = \langle J(t), J(t)\rangle : (a, b) \to \mathbb{R}_{\geq 0}$$

is concave up (i.e., has everywhere non-negative second derivative). If so, then $J(0) = J(1) = 0$ implies that $J(t) = 0$ for all t. Hence the Lemma above tells us that there could be no singularities in exp.

To prove concave-up-ness, observe:

$$\begin{aligned} \frac{d^2}{dt}\langle J(t), J(t)\rangle &= \frac{d}{dt} 2\langle \nabla_{\partial_t} J(t), J(t)\rangle \\ &= 2\langle \nabla_{\partial_t}\nabla_{\partial_t} J(t), J(t)\rangle + 2\langle \nabla_{\partial_t} J(t), \nabla_{\partial_t} J(t)\rangle. \end{aligned}$$

By the Jacobi field condition, the acceleration term is replaced by a multiple of the sectional curvature—explicitly, since

$$K(\dot\gamma(t), J(t)) = \frac{(\Omega(J(t), \dot\gamma(t))\dot\gamma(t), J(t))}{|J(t)|^2|\dot\gamma|^2 - (J(t), \dot\gamma(t))^2}$$

we have

$$\frac{d^2}{dt}\langle J(t), J(t)\rangle = 2\left(|J(t)|^2|\dot\gamma|^2 - (J(t), \dot\gamma(t))^2\right)(-K(\dot\gamma(t), J(t))) + 2\langle \nabla_{\partial_t} J(t), \nabla_{\partial_t} J(t)\rangle.$$

The righthand side is a sum of two non-negative terms thanks to the curvature hypothesis, and we are finished. $\square$