Lecture 34: Hadamaard's Theorem and Covering spaces

In the next few lectures, we'll prove the following statement:

Theorem 34.1. Let (M, g) be a complete Riemannian manifold. Assume that for every $p \in M$ and for every $x, y \in T_pM$, we have

$$K(x, y) \leq 0.$$

Then for any $p \in M$, the exponential map

$$\exp_p : T_pM \to M$$

is a local diffeomorphism, and a covering map.

1. Complete Riemannian manifolds

Recall that a connected Riemannian manifold (M, g) is *complete* if the geodesic equation has a solution for all time t. This precludes examples such as $\mathbb{R}^2 - \{0\}$ with the standard Euclidean metric.

2. Covering spaces

This, not everybody is familiar with, so let me give a quick introduction to covering spaces.

Definition 34.2. Let $p : \tilde{M} \to M$ be a continuous map between topological spaces. p is said to be a *covering map* if

(1) p is a local homeomorphism, and
(2) p is *evenly covered*—that is, for every $x \in M$, there exists a neighborhood $U \subset M$ so that $p^{-1}(U)$ is a disjoint union of spaces $\tilde{U}_\alpha$ for which $p|_{\tilde{U}_\alpha}$ is a homeomorphism onto U.

Example 34.3. Here are some standard examples:

(1) The identity map $M \to M$ is a covering map.
(2) The obvious map $M \coprod \ldots \coprod M \to M$ is a covering map.
(3) The exponential map $t \mapsto \exp(it)$ from $\mathbb{R}$ to S^1 is a covering map. First, it is clearly a local homeomorphism—a small enough open interval in $\mathbb{R}$ is mapped homeomorphically onto an open interval in S^1.

As for the evenly covered property—if you choose a proper open interval inside S^1, then its preimage under the exponential map is a disjoint union of open intervals in $\mathbb{R}$, all homeomorphic to the open interval in S^1 via the exponential map.

Non-Example 34.4. The inclusion $\mathbb{R}^2 - \{0\} \hookrightarrow \mathbb{R}^2$ is not a covering map—it is a local homeomorphism, but it is not even surjective, so does not satisfy the evenly covered property. Or, the map $\mathbb{R}^2 \coprod (\mathbb{R}^2 - \{0\}) \to \mathbb{R}^2$ is not a covering map, though it is a local homeomorphism . The map is not evenly covered at the origin of $\mathbb{R}^2$.

2.1. Some facts about covering spaces.

Proposition 34.5. Let $p : \tilde{M} \to M$ be a covering map. Then for every continuous path $\gamma : [0,1] \to M$ and every point $\tilde{q}$ for which $p(\tilde{q}) = \gamma(0)$, there exists a unique path $\tilde{\gamma} : [0,1] \to \tilde{M}$ for which $p\tilde{\gamma} = \gamma$.

This is the path-lifting, or unique path-lifting, property. Moreover,

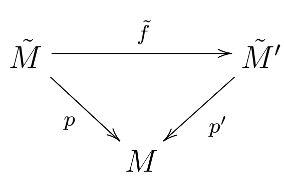
Proposition 34.6. If M is a manifold, then a local diffeomorphism $p : \tilde{M} \to M$ is evenly covered if and only if it satisfies the unique path-lifting property.

Example 34.7. A vector bundle $p : E \to M$ is *not* a covering space. For instance, E and M are manifolds of different dimension, so there is no local homeomorphism from one to the other. But vector bundles do have a (non-unique) path-lifting: Any path in M, on you can lift to a path in E. (Cover the path in M by trivializing neighborhoods—you can assume there are finitely many by compactness of $[0,1]$.)

Theorem 34.8. Given any reasonable space M, there exists a space $\tilde{M}$ and a continuous map $p : \tilde{M} \to M$ such that

(1) $\tilde{M}$ is simply-connected—that is, so $\pi_1(M, x_0) = 0$ for all $x_0 \in M$, and
(2) p is a covering map.

Moreover, $p : \tilde{M} \to M$ is unique up to homeomorphism respecting the covering map: If $p' : \tilde{M}' \to M$ satisfies the properties above, there is a homeomorphism $\tilde{f}$ making the following diagram commute:



Chit-chat 34.9. This $\tilde{M}$ is called the *universal cover* of M, and you should think of it as a space that replaces M by one with no π_1. The fundamental group is sometimes the most complicated part of a space, so removing it simplifies matters greatly.

3. Higher homotopy groups

Just as based maps of loops into M form a group, based maps of spheres into M form a group as well. Define

$$\pi_n(M, x_0) = \{\text{Maps } (S^n, *) \to (M, x_0)\}/\text{homotopies fixing } *.$$

What's the group composition? Well, a map from S^n sending $*$ to x_0 is the same thing as a map from the n-cube I^n to M sending ∂I^n to x_0. And you can compose two such maps from I^n into M by gluing the maps along a face on the boundary of I^n.

Chit-chat 34.10. The homotopy groups $\pi_n(M, x_0)$ are in some sense the mother of all invariants in homotopy theory. For example, we have:

Theorem 34.11 (Whitehead's Theorem)**.** If M and N are reasonable spaces (CW complexes, for instance) and if $f : M \to N$ is a continuous map inducing isomorphisms on π_n for all n, then f admits a homotopy inverse. That is, M and N are homotopy equivalent spaces.

Chit-chat 34.12. But the π_n are notoriously difficult to compute—for instance, we don't know them for spheres.

Proposition 34.13. Let $p : \tilde{M} \to M$ be a covering map. Then for any p induces an isomorphism on π_n for all $n \geq 2$.

Chit-chat 34.14. This is proved easily using, for instance, the long exact sequence for a fibration, or simply by using the unique path-lifting property.

4. Discussion of main theorem

So what are some consequences of the main theorem we stated in this lecture (at the very beginning)?

Corollary 34.15 (Hadamaard's Theorem)**.** Let (M, g) be as in the hypothesis above. If M is simply-connected, then M is diffeomorphic to $\mathbb{R}^{\dim M}$.

PROOF. If M is simply-connected, the identity map exhibits M as its own universal cover. On the other hand, $\exp_p : T_pM \to M$ exhibits T_pM as the universal cover. By uniqueness of universal cover, this means we have a continuous inverse to $\exp_p$. Since $\exp_p$ is also a local diffeomorphism, this inverse map is also smooth—hence $\exp_p$ is a diffeomorphism. $\square$

Corollary 34.16. Let (M, g) be as in the hypothesis. Then all the higher homotopy groups of M vanish.

Chit-chat 34.17. This puts a huge restriction on the spaces for which we can put metrics with non-positive curvature. In fact, the easiest way to cook up manifolds with constant curvature, for instance, is to mod out one good example by properly discontinuous group actions.

5. Toward a proof

So let's get started on the proof. We need some techniques.

Fix $p \in M$ and consider a smooth map

$$w : (\epsilon, \epsilon) \to T_p M.$$

That is, a path of tangent vectors $w(s)$. For each $w(s)$, let's consider the geodesic from p with tangent vector $w(s)$. This gives us a map

$$f : (\epsilon, \epsilon) \times (a, b) \to M, \qquad \gamma_{w(s)}(t)$$

where $\gamma_{w(s)}$ is the geodesic in the direction $w(s)$. Since

$$\gamma_w(t) = \gamma_{tw}(1) = \exp_p(tw)$$

one can write f as a composite of two maps:

$$(-\epsilon, \epsilon) \times (a, b) \xrightarrow{tw(s)} T_p M \xrightarrow{\exp_p} M.$$

Chit-chat 34.18. If we measure the derivative of f in the t direction, we just get the velocity vectors of the geodesic. But if we measure it in the s direction, we see how points in M are changed as we change the initial tangent vector of a geodesic. At $t = 0$ you can see from the equation that the s derivative is zero, but as time goes on, we can measure how quickly (or slowly) the geodesics $\gamma_{w(s)}$ deviate from one another. Moreover, this rate of change satisfies a differential equation governed chiefly by curvature:

Proposition 34.19. Let

$$J(t) = \left.\frac{\partial}{\partial s}\right|_{(0,t)} f.$$

Then J satisfies the differential equation

$$\nabla_{\partial_t} \nabla_{\partial_t} J = \Omega(\dot\gamma(t), J(t))\dot\gamma(t).$$

We'll get to a proof of this next lecture.