Last time:

Flows: Given $X \in \Gamma(TM)$, locally have

$$\Phi^X : (-\varepsilon, \varepsilon) \times U \longrightarrow M$$

so that
- $\Phi^X(0,p) = p$
- $\frac{\partial}{\partial t} \Phi^X(t,p) = X(\Phi^X(t,p))$.

This gave rise to

- Lie derivative $\mathcal{L}_X$ —
- Interpretation of $[X,Y]$.

$$p : \mathbb{R}^2 \longrightarrow M$$
$$(s,t) \longmapsto \Phi^Y_t \Phi^X_s \Phi^Y_{-t} \Phi^X_{-s}(p)$$

$$\frac{\partial^2 p}{\partial s \partial t} = [X,Y].$$

Today, we'll interpret curvature, then define exponential map.

<u>Thm</u> Let $E \to M$ be a smth vec bundle, and fix

- $X, Y \in \Gamma(TM)$
- $\mathcal{H} \subset TE$ an Ehresmann connection.

The following functions

$$\Gamma(E) \longrightarrow \Gamma(E)$$

are all equal:

(1) $D \circ \nabla^{\mathcal{H}}$ — algebraically defined — extending Leibniz rule.

(2) $\nabla^{\mathcal{H}}_X \nabla^{\mathcal{H}}_Y - \nabla^{\mathcal{H}}_Y \nabla^{\mathcal{H}}_X - \nabla^{\mathcal{H}}_{[X,Y]}$ — Also algebraic. Is $\Gamma(TM) \to \mathrm{End}_{\mathbb{R}}(\Gamma(E))$ a map of Lie algebras?

(3) $S \longmapsto \underbrace{[\tilde{X}, \tilde{Y}]_V}\circ S$

$\tilde{X}, \tilde{Y}$ lifts of X, Y to $\mathcal{H}$. Take their Lie bracket, and project it to the vertical component. Since $V \cong \pi^*E$,

$$\begin{tikzcd} & \pi^*E \cong V \\ M \arrow[r, "S"] & E \arrow[u, "{[\tilde{X},\tilde{Y}]_V}"'] \end{tikzcd}$$

yields a new section of E.

<u>Pf</u> (1) $\Leftrightarrow$ (2). Compute in local frame.

For (1) <=> (3), we'll want a lemma:

Lemma Fix $X \in \Gamma(TM)$, $\underline{s} \in \Gamma(E)$.

Let $\widetilde{\Phi}$ be flow of $\widetilde{X}$. Then

$$\nabla^{\mathcal{A}}_X \underline{s} = \lim_{t \to 0} \frac{(\widetilde{\Phi}_t)^* \left(s(\phi_t(p)) \right) - s(p)}{t}$$

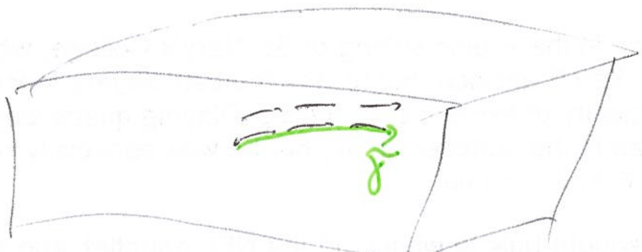
section of E

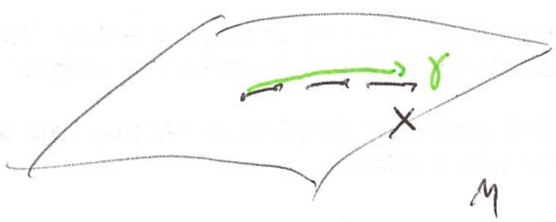
for every $p \in M$, get an element of $T_{s(p)}(E_p) \cong E_p$.

$$T_{s(p)}(E_p) \cong V_{s(p)}$$

$$= \frac{\partial}{\partial t}\bigg|_{t=0} (\widetilde{\Phi}_t)^* (s(\phi_t(p))).$$

Rmk If $\tilde{\gamma}: \mathbb{R} \to E$ is an integral curve for $\tilde{X}$, then $\gamma = \pi \circ \tilde{\gamma}$ is an integral curve for X.





For

$$\begin{aligned} T\gamma\Big|_{t=t_0}\left(\frac{\partial}{\partial t}\right) &= T(\pi \circ \tilde{\gamma})\Big|_{t=t_0}\left(\frac{\partial}{\partial t}\right) \\ &= T\pi\Big|_{\tilde{\gamma}(t_0)} \circ T\tilde{\gamma}\Big|_{t=t_0}\left(\frac{\partial}{\partial t}\right) \\ &= T\pi\Big|_{\tilde{\gamma}(t_0)} \tilde{X}(\tilde{\gamma}(t_0)) \\ &= X(\gamma(t_0)). \end{aligned}$$

— Recall

$$\mathcal{H} \hookrightarrow \mathcal{H} \oplus \mathcal{V} \cong TE \xrightarrow{\text{"}T\pi\text{"}} \pi^* TM$$

is an isomorphism.

Pf of Lemma: Consider

$$\mathbb{R}\times E \xleftarrow{id_{\mathbb{R}}\times\tilde{\Phi}_{-t}} \mathbb{R}\times E \xleftarrow{id_{\mathbb{R}}\times s} \mathbb{R}\times M \xleftarrow{id_{\mathbb{R}}\times\phi_t} \mathbb{R}\times M$$

The derivatives of each of these are the matrices

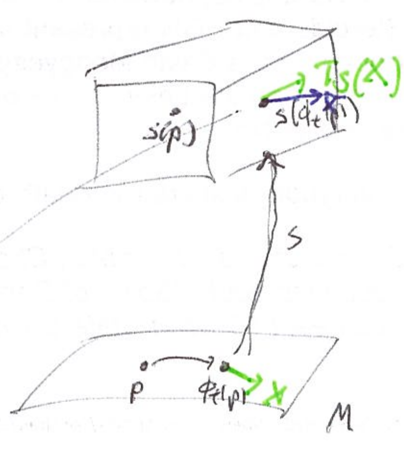
$$\left(\begin{array}{c|c} 1 & 0 \\ \hline \underbrace{-\tilde{X}(\cancel{\quad})}_{-\tilde{X}(\tilde{\Phi}_{-t}\circ s\circ\phi_t(p))} & \underbrace{T\tilde{\Phi}_{-t}\big|_{s(\phi_t(p))}}_{\text{since } (\phi_t)^* = (\phi_{-t})_*} \end{array}\right) \quad \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Ts\big|_{\phi_t(p)} \end{array}\right) \quad \left(\begin{array}{c|c} 1 & 0 \\ \hline X(\phi_t(p)) & T\phi_t\big|_p \end{array}\right)$$

Then $\frac{\partial}{\partial t} = \begin{pmatrix}1\\0\end{pmatrix} \in \begin{array}{c} T_t\mathbb{R} \\ \oplus \\ T_pM \end{array}$ is sent to?

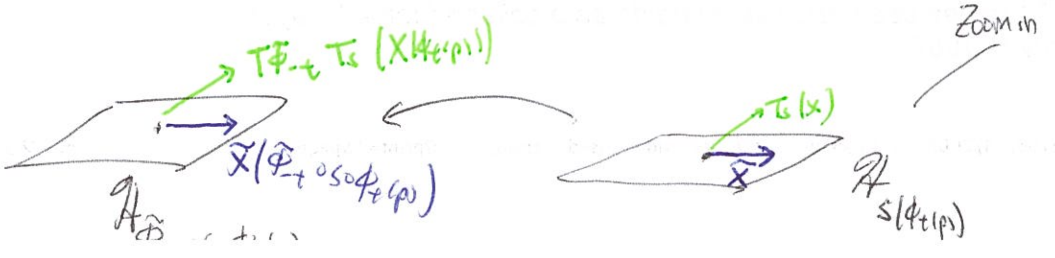
$$T\tilde{\Phi}_{-t}\circ Ts(X(\phi_t(p))) \underbrace{- \tilde{X}(\cancel{\quad})}_{\tilde{\Phi}_{-t}\circ s\circ\phi_t(p)} \longleftarrow Ts\big|_{\phi_t(p)} X(\phi_t(p)) \longleftarrow X(\phi_t(p)) \longleftarrow \begin{pmatrix}1\\0\end{pmatrix}$$

So the limit on righthand side of Lemma is

$$T\tilde{\Phi}_{-t}\circ Ts\,(X(\phi_t(p))) - \tilde{X}\Big(\overset{\tilde{\Phi}_{-t}\circ s\circ\phi_t(p)}{\cancel{s(\phi_t(p))}}\Big) \quad @\ t=0.$$



Zoom in



At $t=0$, this becomes

$$T_s(X(p)) - \tilde{X}(s(p)).$$

OTOH, the left hand side is defined by saying: "Take T_s of X, and take the vertical component":

$$\begin{tikzcd}
TM \arrow[r, "T_s"] \arrow[drrrr, "\nabla^{\mathcal{H}} s"'] & TE \cong \mathcal{H} \oplus V \arrow[r] & V \cong \pi^* E \arrow[r, hook] & & E \times E_{fiber} \arrow[d, "p_{fiber}"] \\
& & & & E
\end{tikzcd}$$

So we just need to show that

$$\tilde{X}(s(p)) = \left(T_s(X(p))\right)_{\mathcal{H}}$$

i.e., that $\tilde{X}(s(p))$ is the horizontal component of $T_s(X(p))$

By the isomorphism

$$\mathcal{H} \longrightarrow \pi^* TM$$

we just need to show that

$$T\pi(\tilde{X}) = T\pi(T_s(X)).$$ i.e. show $T_s(X) - \tilde{X} \in \ker(T\pi)$.

But this is obvious — $\tilde{X}$ is defined so

$$T\pi(\tilde{X}) = X$$

while

$$T\pi(Ts(X)) = T(\pi \circ s)(X)$$

$$= T(id)X$$

$$= X. \quad //$$

We omit proof of (1) <=> (3). Given Lemma, just examine

$$\tilde{P} : \mathbb{R}^2 \longrightarrow E$$

$$(s,t) \longmapsto \tilde{\Psi}_t \tilde{\Phi}_s \tilde{\Psi}_{-t} \tilde{\Phi}_{-s}(s(p))$$

where $\tilde{\Phi}$ is flow of $\tilde{X}$

$\tilde{\Psi}$ is flow of $\tilde{Y}$.

Geodesics.

Let N be a smth mfld, $E \to M$ a vec bundle.

Consider a smooth map $\tilde{f}: N \to E$, so we have a diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ N \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}.$$

<u>Defn</u> Given $\mathcal{H} \subset TE$, we say $\tilde{f}$ is <u>parallel</u> if

$$T\tilde{f}(TN) \subset \mathcal{H}.$$

If $N = U \subset \mathbb{R}$ is an open interval, we say $\tilde{f}$ is parallel along f.

In light of the lemma, this means

$$\nabla_X \tilde{f} = 0$$

$\forall \quad X \in \Gamma(TN)$.

Rmk If $E = TM$, any $f: N \to M$ gives rise to a section

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ N \arrow[ur, "\tilde{f}_X"] \arrow[r, "f"'] & M \end{tikzcd}$$

given a vector field $X \in \Gamma(TN)$:

$$\begin{tikzcd} TN \arrow[r, "Tf"] & TM \\ N \arrow[u, "X"] \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

If $N = \mathbb{R}$, we have a natural vector field

$$\frac{\partial}{\partial t} \in \Gamma(T\mathbb{R}).$$

<u>**Defn**</u>

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$$

is called a <u>geodesic</u> if

$\dot{\gamma}$ is parallel along γ.

$$\begin{tikzcd}
T(-\varepsilon,\varepsilon) \arrow[r, "T\gamma"] & TM \arrow[d, "\pi"] \\
(-\varepsilon,\varepsilon) \arrow[u, "\frac{\partial}{\partial t}"] \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"'] & M
\end{tikzcd}$$

<u>**Rmk**</u> By linearity, this is equivalent to

$$\nabla_{\frac{\partial}{\partial t}} \dot{\gamma} = 0.$$