

Last time:

Flows : Given $X \in \Gamma(TM)$, locally have

$$\Phi^X: (-\varepsilon, \varepsilon) \times U \rightarrow M$$

so that • $\Phi^X(0, p) = p$

• $\frac{\partial}{\partial t} \Phi^X(t, p) = X(\Phi^X(t, p)).$

This gave rise to

• Lie derivative $L_X -$

• Interpretation of $[X, Y]$.

$$P: \mathbb{R}^2 \rightarrow M$$

$$(s, t) \mapsto \Phi_t^{-1} \Phi^X_s \Phi^Y_{-t} \Phi^X_s(p)$$

$$\frac{\partial^2 P}{\partial s \partial t} = [X, Y].$$

Today, we'll interpret curvature, then define exponential map.

Thm let $E \rightarrow M$ be a smth vec bundle,
and fix

- $X, Y \in \Gamma(TM)$

- ∇^H a Ehresmann connection.

The following functions

$$\Gamma(E) \longrightarrow \Gamma(E)$$

are all equal:

$$(1) \quad D \circ \nabla^H$$

algebraically defined - extending Leibniz rule.

$$(2) \quad \nabla_x^H \nabla_y^H - \nabla_y^H \nabla_x^H - \nabla_{[X,Y]}^H$$

Also algebraic. Is
 $\Gamma(TM) \rightarrow \text{End}_{\mathbb{R}}(\Gamma(E))$
a map of Lie algebras?

$$(3) \quad S \longmapsto \underbrace{[\tilde{X}, \tilde{Y}]_V}_{} \circ S$$

\tilde{X}, \tilde{Y} lifts of X, Y to H .

Take their Lie bracket,
and project it to the vertical
component. Since $V \cong \pi^* E$,
 $\pi^* E \cong V$

$$M \xrightarrow{S} E \xrightarrow{[\tilde{X}, \tilde{Y}]_V} [X, Y]_V$$

gives a new section of E .

PF (1) \Leftrightarrow (2). Compute in local frame.

So far we have seen that if \mathcal{F} is a foliation of M , then \mathcal{F} is a submanifold of M . Now we want to show that \mathcal{F} is a submanifold of M if and only if \mathcal{F} is a foliation of M .

For $(1) \Leftrightarrow (3)$, we'll want a lemma:

Lemma Fix $X \in \Gamma(TM)$, $s \in \Gamma(E)$.

Let $\tilde{\Phi}$ be flow of X . Then

$$\nabla_X s = \lim_{t \rightarrow 0} \frac{(\tilde{\Phi}_t)^*(s|_{\tilde{\Phi}_t(p)}) - s(p)}{t}$$

Section of E

for every $p \in M$, get

an element of $T_{s(p)}(E_p) \cong E_p$.

$$s|_{\tilde{\Phi}_t(p)}$$

$$= \left. \frac{\partial}{\partial t} \right|_{t=0} (\tilde{\Phi}_t)^*(s|_{\tilde{\Phi}_t(p)})$$

Given $s \in \Gamma(E)$, we want to show that $\nabla_X s = 0$ if and only if s is a section of E . To do this, we will use the fact that $\tilde{\Phi}_t$ is a diffeomorphism for all $t \neq 0$. This follows from the fact that $\tilde{\Phi}_t$ is a local diffeomorphism and the fact that $\tilde{\Phi}_0 = \text{id}_M$.

So, if $\nabla_X s = 0$, then s is a section of E . Conversely, if s is a section of E , then $s|_{\tilde{\Phi}_t(p)} = s(p)$ for all $t \neq 0$.

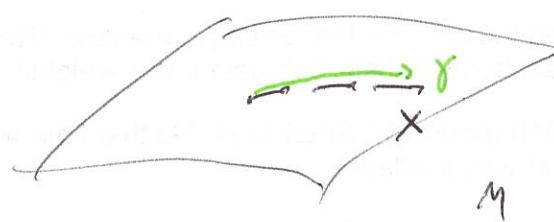
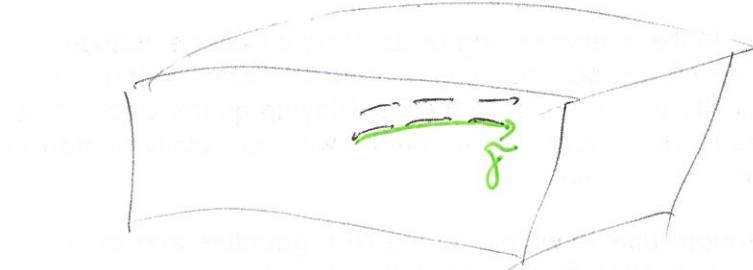
Now we want to show that s is a section of E if and only if s is a section of E . To do this, we will use the fact that $\tilde{\Phi}_t$ is a diffeomorphism for all $t \neq 0$.

So, if s is a section of E , then $s|_{\tilde{\Phi}_t(p)} = s(p)$ for all $t \neq 0$. Conversely, if $s|_{\tilde{\Phi}_t(p)} = s(p)$ for all $t \neq 0$, then s is a section of E .

Rmk If $\tilde{\gamma}: \mathbb{R} \rightarrow E$ is an integral curve for \tilde{X} , then

$\gamma = \pi \circ \tilde{\gamma}$

is an integral curve for X .



For

$$T\tilde{\gamma}\Big|_{t=t_0}\left(\frac{d}{dt}\right) = T(\pi \circ \tilde{\gamma})\Big|_{t=t_0}\left(\frac{d}{dt}\right)$$

$$= T\pi\Big|_{\tilde{\gamma}(t_0)} \circ T\tilde{\gamma}\Big|_{t=t_0}\left(\frac{d}{dt}\right)$$

$$= T\pi\Big|_{\tilde{\gamma}(t_0)} \tilde{X}(\tilde{\gamma}|_{t_0}) \quad \text{— Recall}$$

$$= X(\gamma|_{t_0}).$$

$$\begin{array}{c} \mathcal{H} \hookrightarrow \mathcal{H} \oplus V \cong TE \xrightarrow{T\pi} \pi^* TM \\ \text{is an isomorphism.} \end{array}$$

Pf of Lemma: Consider

$$\mathbb{R} \times E \xleftarrow{\text{id}_{\mathbb{R}} \times \Phi_t} \mathbb{R} \times E \xleftarrow{\text{id}_{\mathbb{R}} \times S} \mathbb{R} \times M \xleftarrow{\text{id}_{\mathbb{R}} \times \phi_t} \mathbb{R} \times M$$

The derivatives of each of these ~~are~~ give the matrices

$$\begin{pmatrix} 1 & 0 \\ -X(\Phi_t \circ S \circ \phi_t(p)) & T\Phi_t \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & T_S \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ X(\phi_t(p)) & T\phi_t \end{pmatrix}$$

$$-X(\Phi_t \circ S \circ \phi_t(p)) \quad \text{since } (\Phi_t)^* = (\phi_t)_*$$

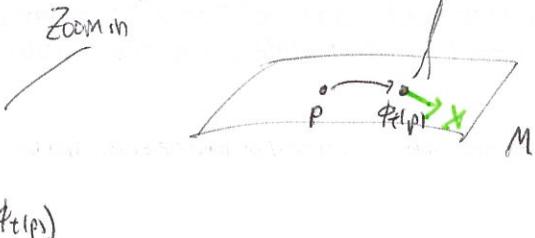
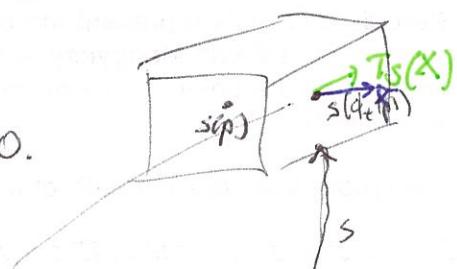
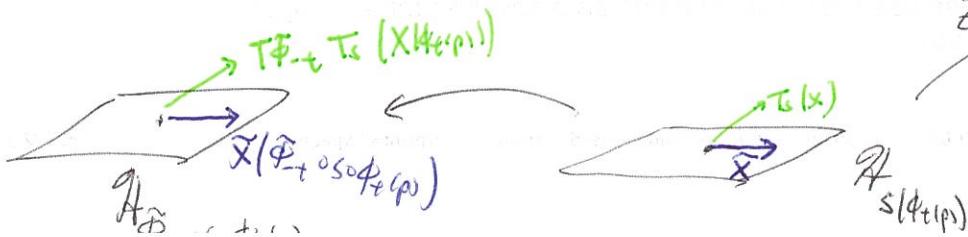
Then $\frac{d}{dt} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \frac{T_t \mathbb{R}}{T_p M}$ be sent to?

$$T\Phi_t \circ T_S(X(\phi_t(p))) \xleftarrow{T_S|_{\Phi_t(p)}} X(\phi_t(p)) \xleftarrow{X|_{\phi_t(p)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$-X(\Phi_t \circ S \circ \phi_t(p))$

So the limit on righthand side of lemma is

$$T\Phi_t \circ T_S(X(\phi_t(p))) - X(\Phi_t \circ S \circ \phi_t(p)) @ t=0.$$



At $t=0$, this becomes

$$T_S(X_{(p)}) = \tilde{X}(s(p)).$$

OTOH, the left hand side is defined by saying: "Take T_S of X , and take the vertical component":

$$\begin{array}{c} TM \xrightarrow{T_S} TE \cong \mathcal{H} \oplus V \xrightarrow{\quad} V \cong \pi^* E \hookrightarrow E \times E_{\text{fiber}} \\ \searrow \nabla^{\mathcal{H}} s \qquad \qquad \qquad \downarrow p_{\text{fiber}} \end{array}$$

So we just need to show that

$$\tilde{X}(s(p)) = (T_S(X_{(p)}))_q$$

i.e., that $\tilde{X}(s(p))$ is the horizontal component of $T_S(X_{(p)})$.

By the isomorphism

$$\mathcal{H} \rightarrow \pi^* TM$$

we just need to show that

$$T_{\pi}(\tilde{X}) = T_{\pi}(T_S(X)).$$

i.e., show $T_S(X) - \tilde{X} \in \ker(T_{\pi})$.

But this is obvious — \tilde{X} is defined so

$$T\pi(\tilde{X}) = X$$

while

$$T\pi(Ts(X)) = T(\pi \circ s)(X)$$

$$= T(id)X$$

$$= X.$$

We omit proof of $(1) \Leftrightarrow (3)$. Given Lemma, just examine

$$\tilde{P}: \mathbb{R}^2 \rightarrow E$$

$$(s, t) \mapsto \mathcal{Y}_t \tilde{\mathcal{P}}_s \mathcal{Y}_t \tilde{\mathcal{P}}_{-s} (s, t)$$

where $\tilde{\mathcal{P}}$ is flow of \tilde{X}

\mathcal{Y} is flow of \tilde{Y} .

Gradesirs.

Let N be a smooth manifold, $E \rightarrow M$ a vec bundle.

Consider a smooth map $\tilde{f}: N \rightarrow E$, so we have a diagram

$$\begin{array}{ccc} & \tilde{f} & \\ N & \xrightarrow{f} & M \\ & \pi & \end{array}$$

Defn Given $\mathcal{A} \subset T^*E$, we say \tilde{f} is

parallel if

$$T\tilde{f}(TN) \subset \mathcal{A}.$$

If $N = U \subset \mathbb{R}$ is an open interval,

we say \tilde{f} is parallel along f .

In light of the lemma, this means

$$\nabla_X \tilde{f} = 0$$

$$\forall X \in \Gamma(TN).$$

Rmk If $E = TM$, any $f: N \rightarrow M$ gives rise to a section

$$\begin{array}{ccc} & TM & \\ f^* \downarrow \pi & \nearrow & \\ N & \xrightarrow{f} & M \end{array}$$

given a vector field $X \in T(N)$:

$$\begin{array}{ccc} TN & \xrightarrow{Tf} & TM \\ X \uparrow & \nearrow f^* & \\ N & & \end{array}$$

If $N = \mathbb{R}$, we have a natural vector field

$$\frac{\partial}{\partial t} \in \Gamma(T\mathbb{R}).$$

Defn: A curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ is called a geodesic if it is locally length-minimizing. This means that for every point $x \in \gamma$, there exists a neighborhood U of x such that the length of any curve $\tilde{\gamma}$ in U connecting $\gamma(t)$ and $\gamma(t')$ is at least as great as the length of $\gamma|_{[t,t']}$.

Defn

$$\gamma: (-\varepsilon, \varepsilon) \rightarrow M$$

is called a geodesic if

γ is parallel along γ .

$$\begin{array}{ccc} T(-\varepsilon, \varepsilon) & \xrightarrow{\text{Tr}} & TM \\ \frac{\partial}{\partial t} \downarrow & \nearrow \gamma & \downarrow \pi \\ (-\varepsilon, \varepsilon) & \xrightarrow{\gamma} & M \end{array}$$

That is, γ is parallel along itself. This means that the tangent vector $\dot{\gamma}(t)$ is parallel to the tangent space $T_{\gamma(t)}M$ for all $t \in (-\varepsilon, \varepsilon)$. In other words, the curve γ is locally length-minimizing.

Rmk By linearity, this is equivalent to

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

Intuitively, this says that the curve γ is "straightest possible" in its neighborhood. If we were to move the curve γ slightly, then the resulting curve would be longer than γ . This is what it means for γ to be locally length-minimizing.

It's also useful to note that if γ is a geodesic, then the velocity vector $\dot{\gamma}(t)$ is parallel transported along γ . This means that if we move the curve γ slightly, then the resulting curve will have the same velocity vector as γ .

Finally, and perhaps most importantly, if γ is a geodesic, then the curve γ is locally length-minimizing. This means that for every point $x \in \gamma$, there exists a neighborhood U of x such that the length of any curve $\tilde{\gamma}$ in U connecting $\gamma(t)$ and $\gamma(t')$ is at least as great as the length of $\gamma|_{[t,t']}$.

That is, γ is parallel along itself. This means that the tangent vector $\dot{\gamma}(t)$ is parallel to the tangent space $T_{\gamma(t)}M$ for all $t \in (-\varepsilon, \varepsilon)$. In other words, the curve γ is locally length-minimizing.

It's also useful to note that if γ is a geodesic, then the velocity vector $\dot{\gamma}(t)$ is parallel transported along γ . This means that if we move the curve γ slightly, then the resulting curve will have the same velocity vector as γ .