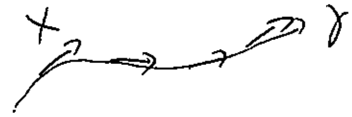
Flows: Let X be a vector field on M.

Thinking of X as a velocity field for some fluid, it's natural to ask for curves

$$\gamma : \mathbb{R} \longrightarrow M$$



or smaller interval

such that $\gamma' = X \circ \gamma$.

Such a γ is called an integral curve.

Thm Let $F : \mathcal{U} \longrightarrow \mathbb{R}^n$, $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$ be a smooth map. Then $\forall\ x \in \mathcal{U}$, $\exists\ \varepsilon > 0$, $V \subset \mathcal{U}$ containing x, and a smooth map

$$\Phi : (-\varepsilon, \varepsilon) \times V \longrightarrow \mathcal{U}$$

such that

$$t \overset{\gamma_x}{\longmapsto} \Phi(t, x)$$

satisfies

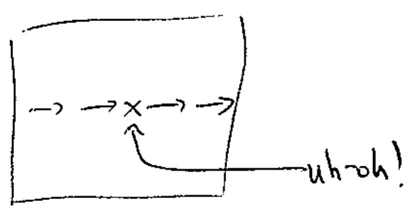
- $\gamma_x(0) = x$
- $\gamma_x'(t) = F(\gamma_x(t))$

$\forall\ (t, x) \in (-\varepsilon, \varepsilon) \times V$. If $\tilde{\Phi}^* : (-\varepsilon', \varepsilon') \times V' \to \mathcal{U}$ is another such map, $\Phi = \tilde{\Phi}^*$ where they are both defined.

Likewise, if $\tilde{\gamma}_x$ is another integral curve, $\tilde{\gamma}_x = \gamma_x$.

What does this mean? Locally, $TU \cong U \times \mathbb{R}^m$, so a vector field is a function $U \to \mathbb{R}^m$. The theorem guarantees we can find an integral curve passing through any $x \in U$; moreover, we can find simultaneous integral curves $\forall\, x \in V$ — i.e., a flow of the whole open set V.

__Rmk__ Φ cannot exist for all t. Consider the vector field $\frac{\partial}{\partial x_1}$ on $\mathbb{R}^2 \setminus \{0\}$.



However:

__Cor__ Let X be a <u>compactly</u> <u>supported</u> vector field on M. Then $\exists!$ smooth map

$$\Phi^X : \mathbb{R} \times M \to M$$

s.t.

$$\gamma_x(t) = \Phi^X(t,x) \text{ satisfies } \gamma_x' = X \circ \gamma_x, \quad \gamma_x(0) = x.$$

__Pf__ Cover support(X) by V_α, where $\exists\ \Phi_\alpha : (-\varepsilon_\alpha, \varepsilon_\alpha) \times V_\alpha \to M$. By compactness, we can choose minimal ε among ε_α. Then we can always flow time $\frac{\varepsilon}{2}$ more, so Φ^X is defined $\forall\ t \in \mathbb{R}$. The flows agree on overlaps by uniqueness. //

<u>Rmk</u> Let $X \in \Gamma(TM)$ be compactly supported. Then the flow

$$\Phi^X : \mathbb{R} \times M \longrightarrow M$$

defines a group homomorphism

$$\Phi^X : \mathbb{R} \longrightarrow \mathrm{Diff}(M)$$

to the group of diffeomorphisms of M:

$$\Phi^X(s+t, p) = \Phi^X(s,-) \circ \Phi^X(t,-)\,(p)$$

because flowing for time t, then flowing for time s, is the same thing as flowing for time $s+t$.

The fact that

$$\Phi^X(t,-) : M \longrightarrow M$$

is a diffeomorphism follows from smoothness and uniqueness of Φ^X — since $\Phi^X(t,-)$ is a smooth bijection, so is

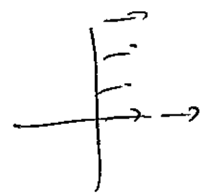
$$\Phi^X(-t,-) = \left(\Phi^X(t,-)\right)^{-1}.$$

Exer Construct a flow

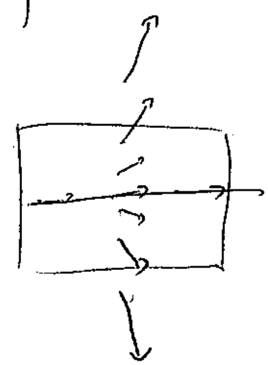
$$\Phi^X : \mathbb{R} \times M \longrightarrow M$$

for $M = \mathbb{R}^2$, and

(a) $X = \partial_{x_1}$



(b) $X(x_1, x_2) = \partial_{x_1} + x_2 \partial_{x_2}$.



Ans

(a) $\Phi^X(t, (x_1, x_2)) = (x_1 + t, x_2)$

(b) $\qquad$ " $\qquad = (x_1 + t, t x_2)$.

What can we do with flows?

(1) Define Lie derivative $\rightsquigarrow$ geom interp. of Lie bracket

(2) Define parallel transport

~~(3) Define~~

(3) Define geodesics

(1)+(2) $\rightsquigarrow$ geometric intepretation of curvature

(3) $\rightsquigarrow$ exponential map

Lie derivatives

Let $\alpha \in \Gamma(TM)$ (or $\alpha \in \Omega^k_{dR}(M)$)

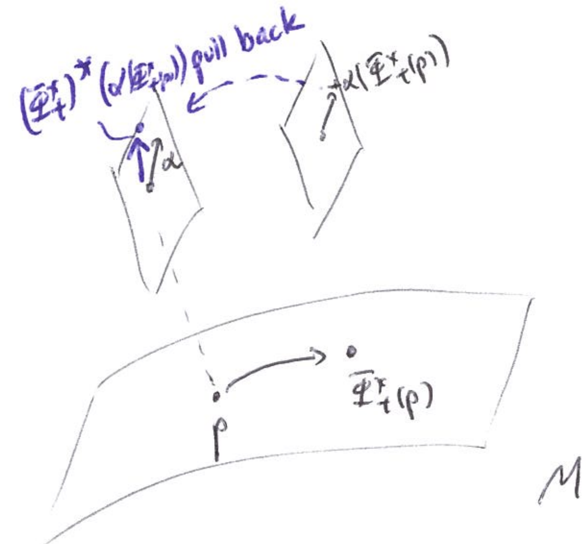
Given $X \in \Gamma(TM)$, we have a diffeomorphism

$$\Phi^X_t : M \to M \quad , \text{or} \quad \Phi^X_t : U \hookrightarrow U'$$

for $t \in (-\varepsilon, \varepsilon)$. Thus we can compare

$$\alpha(p) \quad \text{and} \quad (\Phi^X_t)^* \alpha(\Phi^X_t(p))$$

for all t.



What's the rate of change of the family

$$(\Phi^X_t)^* (\alpha(\Phi^X_t(p))) \quad \in T_pM \quad (\text{or } \Lambda^k T_pM^*)$$

?

Defn The Lie derivative of α with respect to X is

$$\lim_{t \to 0} \frac{(\Phi^Y_t)^* \alpha(\Phi^Y_t(p)) - \overset{(\Phi^Y_0)^* \alpha(\Phi^Y_0(p))}{\overset{\parallel}{\alpha(p)}}}{t}.$$

Thm Let $X, Y \in \Gamma(TM)$. The following vector fields are equivalent:

(1) $[X, Y]$

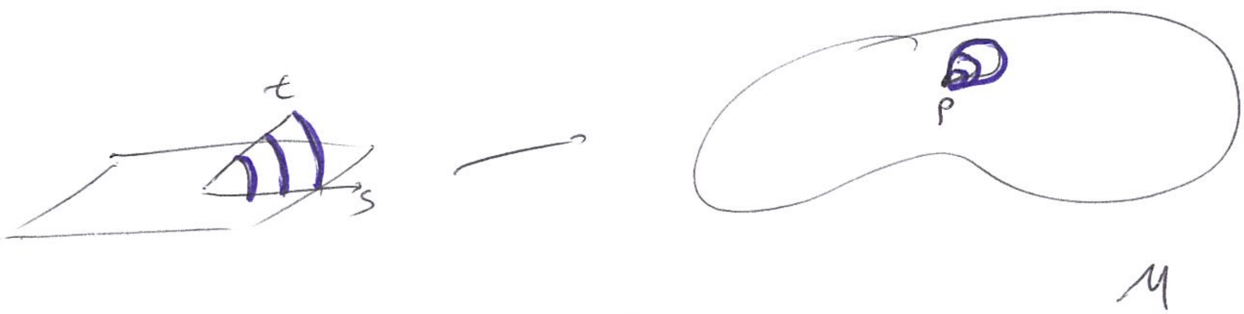
(2) $\mathcal{L}_X Y$

(3) $p \longmapsto \frac{\partial}{\partial t}\Big|_{t=0} \Phi^Y_{\sqrt{t}} \Phi^X_{\sqrt{t}} \Phi^Y_{\sqrt{t}} \Phi^X_{-\sqrt{t}}$.

<u>Rmk</u> (3) looks crazy at first. Here's what's going on:

Consider, $\forall\, p \in M$, the map

$$\begin{array}{ccc} P: \cancel{\mathbb{R}\times\mathbb{R}} & \longmapsto & \psi_t \varphi_s \psi_{-t} \varphi_{-s}(p) \\ (s,t) & & \\ \mathbb{R}^2 & \longrightarrow & M. \end{array}$$



We'll prove

$$\mathcal{L}_X Y = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(s,t)=(0,0)} P .$$

The trick

$$\begin{array}{ccccccc} \mathbb{R}_{\geq 0} & \xrightarrow{\sqrt{t}} & \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \xrightarrow{P} & M \\ t & \longmapsto & \sqrt{t} & \longmapsto & (\sqrt{t},\sqrt{t}) & \longmapsto & P(\sqrt{t},\sqrt{t}) \end{array}$$

turns $\frac{\partial^2}{\partial s \partial t}$ into a first derivative.

$$\mathbb{R}^2\times U \xleftarrow{id_{\mathbb{R}^2}\times\varphi_s} \mathbb{R}^2\times U \xleftarrow{id_{\mathbb{R}^2}\times\psi_t} \mathbb{R}^2\times U \xleftarrow{id_{\mathbb{R}^2}\times\varphi_{-s}} \mathbb{R}^2\times U \xleftarrow{id_{\mathbb{R}^2}\times\psi_{-t}} \mathbb{R}^2\times U$$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \\ \hline X(\varphi_s\psi_t\varphi_{-s}\psi_{-t}(p)) & 0 & T\varphi_s\big|_{\psi_t\varphi_{-s}\psi_{-t}(p)} \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline 0 & Y(\psi_t\varphi_{-s}\psi_{-t}(p)) & T\psi_t\big|_{\varphi_{-s}\psi_{-t}(p)} \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline -X(\varphi_{-s}\psi_{-t}(p)) & 0 & T\Phi_{-s}\big|_{\psi_{-t}(p)} \end{array}\right) \begin{array}{c} \partial_s \;\; \partial_t \\ \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline 0 & \boxed{} & T\psi_{-t}\big|_p \end{array}\right) \\ -Y(\psi_{-t}(p)) \end{array}$$

$$\Big\downarrow \; @ \; (s,t) = (0,p).$$

$$\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline X(p) & 0 & T\varphi_s\big|_p \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline 0 & Y(p) & T\psi_t\big|_p \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline -X(p) & 0 & T\Phi_{-s}\big|_p \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0 \\ \hline 0 & -Y(p) & T\psi_{-t}\big|_p \end{array}\right)$$

So $\frac{\partial}{\partial s}\Big|_{s=0} \beta(0,t) = ?$ Note along $s=0$, $\varphi_s = \varphi_{-s} = \mathrm{id}_M$. So

multiplying matrices,

$$\begin{pmatrix} 1 \\ 0 \\ \hline X(p) - T\psi_t\big|_{\psi_{-t}(p)}\big(X(\psi_{-t}(p))\big) \end{pmatrix} \mapsfrom \begin{pmatrix} 1 \\ 0 \\ \hline -T\psi_t\big|_{\psi_{-t}(p)}\big(X(\psi_{-t}(p))\big) \end{pmatrix} \mapsfrom \begin{pmatrix} 1 \\ 0 \\ \hline -X(\psi_{-t}(p)) \end{pmatrix} \mapsfrom \begin{pmatrix} 1 \\ 0 \\ \hline 0 \end{pmatrix} \mapsfrom \begin{pmatrix} 1 \\ 0 \\ \hline 0 \end{pmatrix}.$$

i.e.,

$$\frac{\partial}{\partial s}\Big|_{s=0} \beta(s,t) = \underbrace{X(p) - T\psi_t\big|_{\psi_{-t}(p)}\big(X(\psi_{-t}(p))\big)}_{\substack{\text{a function of } t, \\ \text{values in } T_pU}}.$$

Likewise, $\frac{\partial}{\partial t}\Big|_{t=0} \beta(s,0)$ is

$$\begin{pmatrix} 0 \\ 1 \\ \hline T\varphi_s\big|_{\varphi_{-s}(p)}\big(Y(\varphi_{-s}(p))\big) - Y(p) \end{pmatrix} \mapsfrom \begin{pmatrix} 0 \\ 1 \\ \hline \underbrace{-T\varphi_{-s}\big|_p Y(p)}_{Y(\varphi_{-s}(p))} \end{pmatrix} \mapsfrom \begin{pmatrix} 0 \\ 1 \\ \hline -T\varphi_{-s}\big|_p Y(p) \end{pmatrix} \mapsfrom \begin{pmatrix} 0 \\ 1 \\ \hline -Y(p) \end{pmatrix} \mapsfrom \begin{pmatrix} 0 \\ 1 \\ \hline 0 \end{pmatrix}$$

i.e.,

$$\frac{\partial}{\partial t}\Big|_{t=0} \beta(s,t) = -Y(p) + T\varphi_s\big|_{\varphi_{-s}(p)}\big(Y(\varphi_{-s}(p))\big).$$

Since $\frac{\partial}{\partial s}\Big|_{s=0}\beta(s,t)$ has no dependence on s,

$$\frac{\partial^2}{\partial s^2}\Big|_{s=0}\beta(s,t)=0.$$

Likewise,
$$\frac{\partial^2}{\partial t^2}\Big|_{t=0}\beta(s,t)=0.$$

So let's compute

$$\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\beta(s,t)=\frac{\partial}{\partial s}\Big|_{s=0}\left(-\underbrace{Y(p)}_{\text{constant in } s}+T\phi_s\big|_{\phi_{-s}(p)}\big(Y(\phi_{-s}(p))\big)\right)$$

$$\mathcal{L}_{-X}Y \longrightarrow =\frac{\partial}{\partial s}\Big|_{s=0}\left(T\phi_s\big|_{\phi_{-s}(p)}Y(\phi_{-s}(p))\right)$$

$$=\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\phi_s\circ\psi_t\circ\phi_{-s}$$

multiply matrices, or precompose β w/ ψ_t $\longrightarrow$

$$=\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}\left(\phi_s\circ\psi_t\circ\phi_{-s}\right)$$

$$=\frac{\partial}{\partial t}\Big|_{t=0}\left(X(\psi_t(p))-T\psi_t\big|_p(X(p))\right)$$

$$=\frac{\partial}{\partial t}\Big|_{t=0}X(\psi_t(p))-\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}\psi_t\phi_s(p)$$

$$=\frac{\partial}{\partial t}\Big|_{t=0}X(\psi_t(p))-\frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial t}\Big|_{t=0}\psi_t\phi_s(p)$$

$$[Y,X]\longrightarrow =\frac{\partial}{\partial t}\Big|_{t=0}X(\psi_t(p))-\frac{\partial}{\partial s}Y(\phi_s(p))$$