Pull-backs

Mon, Nov 10, 2014

Prop. Fix a smth mfld M

and smth maps

$$f_i : L_i \to M, \quad i=1,2.$$

If $f_1 \pitchfork f_2$, then

(1) $L_1 \times_M L_2 := \{(l_1, l_2) \text{ s.t. } f_1(l_1) = f_2(l_2)\}$
is a smth mfld, and

(2) For any smth mfld L' and smth maps

$$g_i : L' \to L_i$$

making

$$\begin{tikzcd}
L' \arrow[dr, "g_2"] \arrow[dd, "g_1"'] & \\
& L_2 \arrow[d, "f_2"] \\
L_1 \arrow[r, "f_1"'] & M
\end{tikzcd}$$

commute, $\exists!$ smth map $L' \xrightarrow{g} L_1 \times_M L_2$

s.t.

$$\begin{tikzcd}
L' \arrow[dr] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& L_1 \times_M L_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & L_2 \\
& L_1 &
\end{tikzcd}$$

commutes

<u>Pf</u> (1) Guillemin and Pollack.

(2) If $f_2 g_2 = f_1 g_1$, let

$$g(\ell) := (g_1(\ell), g_2(\ell)).$$

$g(\ell) \in L_1 \times_M L_2$ by definition. It's C^∞ because each component

$$\begin{aligned} \pi_i \circ g : L' &\longrightarrow L_i \\ \ell &\longmapsto g_i(\ell) \end{aligned} \qquad (*)$$

is C^∞ by assumption. Details: g is C^∞ iff $\forall$ smooth $h : L_1 \times_M L_2 \to \mathbb{R}$, $h \circ g$ is C^∞. By def'n, h is C^∞ iff $\exists\ \tilde{h} : \mathcal{U} \to \mathbb{R}$ smth w/ $L_1 \times_M L_2 \subset \mathcal{U} \overset{\text{open}}{\subset} M$, $\tilde{h}|_{L_1 \times_M L_2} = h$.

$$\begin{tikzcd}
L' \arrow[r, "g"] \arrow[rr, bend left, "\tilde{g}"] & L_1 \times_M L_2 \arrow[r, hook] \arrow[dr, "h"'] & \mathcal{U} \arrow[d, "\tilde{h}"] \\
& & \mathbb{R}
\end{tikzcd}$$

since $\tilde{g}$ is C^∞ by $(*)$, so is hg. //

<u>Ex</u> Pullback bundles:

If

$$f: N \to M \text{ is } C^\infty$$

$$\pi: E \to M \quad \text{a } C^\infty \text{ vec bundle},$$

π is a submersion, so

$$f \pitchfork \pi.$$

Then
$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
N \arrow[r] & M
\end{tikzcd}$$

is the pullback bundle.

Note any section

$$s: M \to E$$

results in a section

$$f^*s: N \to f^*E$$

by
$$\begin{tikzcd}
N \arrow[dr, "f^*s\ \exists!"] \arrow[drr, bend left, "s\circ f"] \arrow[ddr, "\mathrm{Id}"'] & & \\
& f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& N \arrow[r, "f"'] & M
\end{tikzcd}$$

The vertical tangent space

Let $\pi : E \to M$ be a vector bundle.

Note $\exists$ map

$$TE \to \pi^* TM$$

by the diagram

$$\begin{array}{ccc} \pi^* TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \xrightarrow[\pi]{} & M \end{array}$$

and

$$\begin{array}{ccc} TE & \xrightarrow{T\pi} & TM \\ \downarrow & & \downarrow \\ E & \xrightarrow[\pi]{} & M. \end{array}$$

We let

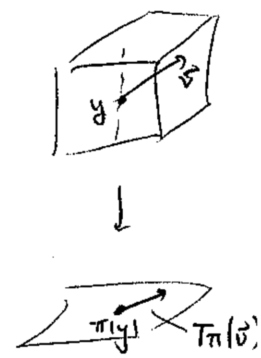
$$V := \operatorname{Ker}(TE \to \pi^* TM)$$

be the vertical tangent bundle of E.

What is this map?

$T\pi : TE \to TM$

says: "Take a tangent vector $\vec{v}$ at $y \in E$, and consider the projected tangent vector $T\pi(\vec{v}) \in TM$."



The map

$$TE \to \pi^* TM$$

says "also remember that you came from y." It sends

$$(y, \vec{v}) \longmapsto (y, T\pi(\vec{v})).$$

<u>Rmk</u> $\exists$ bundle $\cong$

$$V \cong \pi^* E$$

tang. space of <u>fiber</u> (not all of E) at y.

fiber containing y

Clearly, $V = \bigcup_{y \in E} T_y\left(E_{\pi(y)}\right)$

$\cong \bigcup_{y \in E} E_{\pi(y)}$

since for any vector space V, $T_y V \cong V$.

$\left(f \mapsto \frac{\partial}{\partial t}\left(f(y+t\vec{v})\right)\right) \leftarrow\!\shortmid \vec{v}$

$\cong \pi^* E$.

You can check in local coords. that this is smooth.

"Moving vertically is the same thing as moving in fibers."

Defn An Ehresmann connection on $E \xrightarrow{\pi} M$ is a choice of subbundle on E

$$\mathcal{H} \subset TE$$

s.t.

(1) The inclusions $\mathcal{H}, V \hookrightarrow TE$ induce an iso

$$\mathcal{H} \oplus V \cong TE$$

splits tangent directions of E into horizontal and vertical.

(2) $\forall\, r \in \mathbb{R}$, consider the scaling map

$$\rho_r : TE \longrightarrow TE$$
$$y \longmapsto ry$$

Then $T\rho_r(\mathcal{H}) = \mathcal{H}$.

i.e.,

$$T\rho_r(\mathcal{H}_y) = \mathcal{H}_{ry}, \ \forall\, y \in E$$

To make a certain construction clear — if

$$\underset{\in\, TE}{\vec{u}} = \underset{\in\, \mathcal{H} \oplus V}{\vec{u}_h + \vec{u}_v}$$

then

$$T\rho_r(\vec{u}) = T\rho_r(\vec{u}_h) + T\rho_r(\vec{u}_v)$$

will still be a decomposition wrt the splitting

<u>Rmk</u> The composite

$$\mathcal{H} \hookrightarrow \mathcal{H} \oplus V \cong TE \longrightarrow \pi^* TM$$

is an injection since $\mathcal{H} \cap \mathrm{Ker} = 0$, and

a surjection since $TE \to \pi^*TM$ is, while $\mathcal{H} \cong TE/V \cong \underset{\substack{\| \\ \text{image}(TE \to \pi^*TM)}}{\pi^*TM}$.

$\Longrightarrow$ $\mathcal{H} \cong \pi^* TM$,

so given any $X \in \Gamma(TM)$, we have

$$\begin{tikzcd}
E \arrow[r, "\tilde{X}"] \arrow[dr, "id"'] \arrow[rr, bend left, "X \circ \pi"] & \pi^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\
& E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

ie, we have a map

$$\begin{array}{ccc} \Gamma(TM) & \longrightarrow & \Gamma(\pi^* TM) \quad \cong \Gamma(\mathcal{H}) \\ X & \longmapsto & (\tilde{X} : E \to \pi^* TM). \\ \big/ & & \\ (X : M \to TM) & & \end{array}$$

This is $C^\infty(M)$-linear, where the module action on $\Gamma(\pi^* TM)$

is via the ring homomorphism

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{\pi^*} & C^\infty(E) \circlearrowright \Gamma(\pi^* TM), \\ f & \longmapsto & f \circ \pi \end{array}$$

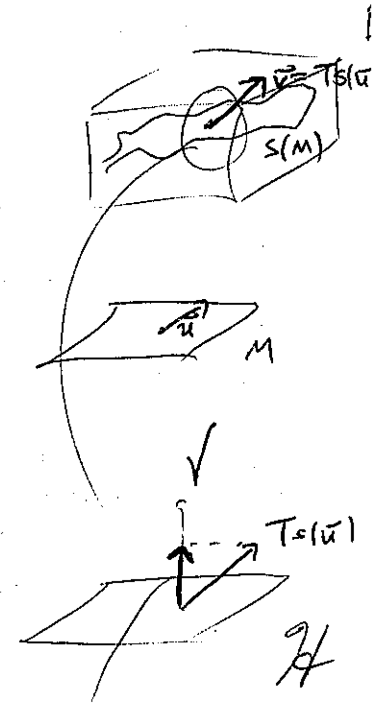
Now, $\forall$ sections $s : M \to E$, consider

$$\begin{tikzcd} TM \arrow[r, "Ts"] \arrow[drrr, "\nabla^{\mathcal{H}} s"'] & TE \cong \mathcal{H} \oplus V \arrow[r] & V \cong \pi^* E \arrow[r, hook] & E_{base} \times E_{fiber} \arrow[d, "fiber"] \\ & & & E \end{tikzcd}$$

every map in this diagram is linear in E_{fiber}, and ~~doesn't~~

hence $TM \to E$ is a smooth bundle map over M — i.e., an element of

$$\Gamma(Hom(TM, E)) \cong \Omega^1_{dR}(M; E).$$



project $Ts(\vec{u})$ to vertical bit.
i.e., "how far is $Ts(\vec{u})$ from being horizontal?"

So we have a map of sets

$$\nabla^{\mathcal{H}} : \Gamma(E) \longrightarrow \Omega^1_{dR}(M; E)$$
$$s \longmapsto \nabla^{\mathcal{H}} s.$$

<u>Prop'n</u> $\nabla^{\mathcal{H}}$ is a connection on E.

Pf (1) Obvious that

$$\nabla^{\mathcal{H}}(s_1+s_2) = \nabla^{\mathcal{H}}s_1 + \nabla^{\mathcal{H}}s_2.$$

Why? Note

$$T(s_1+s_2) = Ts_1 + Ts_2 : TM \to TE$$

and the rest of the diagram is linear.

(2) Leibniz rule: In local coordinates,

$$s|_U : U \longrightarrow E|_U \cong U \times \mathbb{R}^k$$

$$f \cdot s|_U : U \longrightarrow U \times \mathbb{R}^k,$$

compute $T(fs|_U)$ we have $df \otimes s|_U + f\, Ts|_U$.

Since $\mathcal{H}$ is scale-invariant, a splitting

$$u = u_{\mathcal{H}} + u_r \in TE_y$$

remains a splitting

$$T_{pr}(u) = T_{pr}(u_{\mathcal{H}}) + T_{pr}(u_r) \in TE_{ry}.$$

~~$\Rightarrow (T(fs)(v))_r = df(v)$~~ You can do the rest! //

__Thm__ The assignment

$$\left\{\begin{array}{c}\text{Ehresmann cnxns}\\ \mathcal{H}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Connections}\\ \nabla\end{array}\right\}$$

$$\mathcal{H} \longmapsto \nabla^{\mathcal{H}}$$

is a bijection.

__Rmk__ What's the inverse? Given ∇ and $s: M \to E$,

consider

"subtract vertical bit."

$$\mathcal{H}_y = \{ \underbrace{Ts(\vec{v})}_{\text{tangent vector of } E} - \underbrace{\nabla_{\vec{v}} s}_{\substack{\text{an element of } E_{\pi(y)},\\ \text{identify w/ element of } V_y.}} \}$$

This splits TE w/ V, and satisfies (2) in defn of Ehresmann cnxn

because of Leibniz rule.