FRI, NOV 7^{th}
Lecture 28

<u>Prop</u> $\forall$ Riemannian metric g on E,
$\exists$ ∇ compatible w/ g.

<u>Pf</u> On small n-hood $U \subset M$, fix orthonormal frame $s_1, \dots, s_k$. Arbitrarily choose ∇ so

$$\nabla s_i = \alpha_{ij} s_j \quad \left(= \sum_{j=1}^{k} \alpha_{ij} \otimes s_j \right)$$

where $\alpha_{ij} = -\alpha_{ji}$. (Recall ∇ is determined by what it does on a frame.) Then

$$d \underbrace{g(s_i, s_j)}_{\text{constant on } U} \Big|_U = 0 \qquad \text{while} \qquad g(\nabla s_i, s_j) + g(s_i, \nabla s_j) = \alpha_{ij} + \alpha_{ji} = 0.$$

Further,

$$d\, g(f s_i, s_j) = df \cdot \delta_{ij} \quad \text{while} \quad g(\nabla(f s_i), s_j) + g(s_i f, \nabla s_j) = df \cdot \delta_{ij} + f \alpha_{ij} + f \alpha_{ji} = df \cdot \delta_{ij}.$$

So this is compatible.

SwissMAP
The Mathematics of Physics
National Centre of Competence in Research

Fixing a partition of unity h_α, ~~we have~~

$$\cancel{dg(s,t) = \sum_\alpha d(h_\alpha g(s,t))}$$

$$\cancel{= \sum_\alpha}$$

define

$$\nabla = \sum h_\alpha \nabla_\alpha,$$

so

$$\nabla s = \sum h_\alpha \cdot \nabla_\alpha (s|_{U_\alpha}).$$

Then

$$g(\nabla s, t) + g(s, \nabla t) = g\left(\sum h_\alpha \nabla_\alpha (s|_{U_\alpha}), t\right) + g\left(s, \sum h_\alpha \nabla_\alpha (t|_{U_\alpha})\right)$$

$$= \sum_\alpha h_\alpha \left(g|_{U_\alpha}(\nabla_\alpha s|_{U_\alpha}, t|_{U_\alpha}) + g|_{U_\alpha}(s|_{U_\alpha}, \nabla_\alpha (t|_{U_\alpha}))\right)$$

$$= \sum_\alpha h_\alpha \, dg(s|_{U_\alpha}, t|_{U_\alpha})$$

$$= dg(s,t). \quad //$$

<u>Lemma</u> Let ω be a k-form. Then

$$d\omega(X_0,\dots,X_k) = \sum_{0\le i\le k} (-1)^i X_i\, \omega(X_0,\dots,\hat{X_i},\dots,X_k)$$

$$+ \sum_{0\le i<j\le k} (-1)^{i+j} \omega([X_i,X_j],X_0,\dots,X_k).$$

<u>Pf</u> Homework. Also, there is an implicit convention being used, to identify $\Lambda^k T^*X$ w/ space of alternating maps $TX^{\otimes k} \to \mathbb{R}$. More on this later. //

<u>Lemma</u> TFAE for ∇ compatible w/ g:

(1) $\forall\ X, Y \in \Gamma(TM)$

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

(2) Let $\bar{\nabla}$ be

$$\begin{array}{ccc} \Gamma(T^*M) & \xrightarrow{\bar{\nabla}} & \Gamma(T^*M \otimes T^*M) \\ g\ \wr\!\wr & & \wr\!\wr \\ \Gamma(TM) & \xrightarrow{\nabla} & \Gamma(TM \otimes T^*M) \end{array}$$

then

$$\begin{array}{ccc} \Gamma(T^*M) & \xrightarrow{\bar{\nabla}} & \Gamma(T^*M \otimes T^*M) \\ & \searrow^{\circlearrowleft}_{d_{dR}} & \downarrow \wedge \\ & & \Gamma(T^*M \wedge T^*M) = \Omega^2 \end{array}$$

<u>Pf</u> We know if $f|_U = g|_U$, then $(\nabla f)|_U = (\nabla g)|_U$, so suffices to check locally.

Let $\{s_i\}$ be orthonormal basis in local coordinates. If

$$s_i \overset{\nabla}{\longmapsto} \alpha_{ij}\, s_j$$

then $\bar{\nabla}$ is given by

$$\begin{tikzcd}
g(s_i,-) \arrow[r, mapsto] \arrow[d, leftrightarrow] & \alpha_{ij}\, g(s_j,-) \arrow[d, leftrightarrow] \\
s_i \arrow[r, mapsto] & \alpha_{ij}\, s_j
\end{tikzcd}$$

so ∇ and $\bar{\nabla}$ have same α_{ij}.

Let $\theta_i = g(s_i,-) \in \Gamma(T^*M|_U)$. Then

$$(2) \Rightarrow d\theta_i = \alpha_{ij} \wedge \theta_j \quad \text{, but}$$

$$d\theta_i(s_k, s_l) = -\theta_i([s_k, s_l]) + s_k(\underbrace{\theta_i(s_l)}_{\text{const}}) - s_l(\underbrace{\theta_i(s_k)}_{\text{const}})$$

$$= -\theta_i([s_k, s_l])$$

while

$$\alpha_{ij} \wedge \theta_j(s_k, s_l) = \alpha_{ij}(s_k)\,\theta_j(s_l) - \alpha_{ij}(s_l)\,\theta_j(s_k)$$

$$= \alpha_{il}(s_k) - \alpha_{ik}(s_l).$$

OTOH,

$$(1) \Rightarrow ? \quad \nabla_{s_k} s_l = \alpha_{li}(s_k)\, s_i$$

$$-\nabla_{s_l} s_k = -\alpha_{ki}(s_l)\, s_i$$

Then s_i component of $\nabla_{s_k} s_l - \nabla_{s_l} s_k - [s_k, s_l]$ is

$$\alpha_{li}(s_k) - \alpha_{ki}(s_l) - \theta_i([s_k, s_l])$$

$$= -\alpha_{il}(s_k) + \alpha_{ik}(s_l) + d\theta_i(s_k, s_l)$$

$$= \left(-\alpha_{ij} \wedge \theta_j + d\theta_i\right)(s_k, s_l)$$

$$\cancel{= -\left(d\theta_i + \alpha_{ij} \wedge \theta_j\right)(s_k, s_l)}$$

i.e. $\nabla_{s_k} s_l - \nabla_{s_l} s_k - [s_k, s_l] = 0 \ \forall\, k, l$ iff

$$\begin{tikzcd} \theta_i \arrow[r, mapsto, "\nabla"] \arrow[dr, "d"'] & +\alpha_{ij} \theta_j \arrow[d, "\wedge"] \\ & \alpha_{ij} \wedge \theta_j \end{tikzcd}$$

commutes //

(Fundamental Thm of Riemannian Geometry)

Thm $\forall$ Riemannian g on TM,

$\exists!\ \nabla$ s.t.

- ∇ compatible w/ g
- $\nabla_X Y - \nabla_Y X - [X, Y] = 0 \quad \forall X, Y.$

Pf. Let $\theta_i = g(s_i, -)$ be orthonormal frame on a small n-hood. We must solve for α_{ij} so that

$$\nabla \theta_i = \alpha_{ij} \otimes \theta_j, \quad \text{and} \quad \alpha_{kj} \wedge \theta_j = d\theta_k$$

while $\alpha_{ij} = -\alpha_{ji}$.

Well, ~~suppose~~ define A_{ijk} so that

$$d\theta_k = A_{ijk}\, \theta_i \wedge \theta_j, \quad \text{so} \quad \alpha_{kj} = A_{ijk}\, \theta_i.$$

Necessarily, $A_{ijk} = A_{jik}$ since $\theta_i \wedge \theta_j = -\theta_j \wedge \theta_i$.

<u>Lemma</u> $\exists!$ B_{ijk} symm in i,j

C_{ijk} skew in j,k

such that

$$A_{ijk} = B_{ijk} + C_{ijk} \,.$$

I.e.,

$$(\mathrm{Sym}^2 V) \otimes V \oplus V \otimes (\Lambda^2 V) \longrightarrow V^{\otimes 3}$$

is a bijection.

<u>Pf</u> Existence:

$$B_{ijk} = \frac{1}{2}\left(A_{ijk} + A_{jik} - A_{kij} - A_{kji} + A_{jki} + A_{ikj}\right)$$

$$C_{ijk} = \frac{1}{2}\left(A_{ijk} - A_{jik} + A_{kij} + A_{kji} - A_{jki} - A_{ikj}\right)$$

Uniqueness: If this has kernel, $\exists$ D_{ijk} which is symm in ij, skew in j,k. Well,

$$D_{ijk} = D_{jik} = -D_{jki} = -D_{kji} = D_{kij} = D_{ikj} = -D_{ijk}$$

$\Longrightarrow D_{ijk} = 0$. // of Lemma.

To complete theorem: Though we only defined ∇ on trivializing neighborhoods, by uniqueness, $\nabla|_{U_\alpha} = \nabla|_{U_\beta}$. So ∇ defines a connection globally. // of Thm.