CHAPTER 27

Lecture 27: Proof of the Gauss-Bonnet-Chern Theorem.

This will be a sketch of a proof, and we will technically only prove it for 2-manifolds. But I hope indicates some geometric tools and techniques. All the proofs will be sketches until the final computation.

Let $\dim M = 2k = n$. Fix a triangulation K on M.

Here is the strategy we will use:

(1) We will construct a vector field Y on M—i.e., a section of TM—whose critical points (i.e., zeroes) are in one-to-one correspondence with the simplices of K.
(2) We will then argue that there exists a connection for which $\nabla Y = 0$ outside of a small neighborhood U_i of each critical point p_i. As a consequence, $Pf(\Omega_\nabla) = 0$ away from these U_i.
(3) Using Stokes's Theorem, we will calculate that $\int_{U_i} eu(\Omega_\nabla) = (-1)^i$ where i is the dimension of the simplex corresponding to U_i.

Lemma 27.1. For each triangulation $K \to M$, there exists a vector field Y' on M such that the critical points of Y' correspond to the barycenters of each simplex. Moreover, about any critical point p that corresponds to the barycenter of an i-simplex, there exists a local chart U so that

$$Y' = (-x_1\partial_1 - \ldots - x_i\partial_i) + x_{i+1}\partial_{i+1} + \ldots x_n\partial_n.$$

Here, ∂_i is shorthand for $\frac{\partial}{\partial x_i}$.

Proof. This is essentially Morse theory. You construct a smooth function on M step by step—the 0-simplices get value 0, the barycenter of the 1-simplex get the value 1, the barycenter of an i-simplex gets the value i, et cetera. If you look at the vector field given by the gradient of this function, you'll find that each barycenter is a critical point—if it's the barycenter of an i-simplex, it has an i-dimensional ball of directions in which the gradient is flowing down.

Then there is a standard argument from Morse theory that a generic function can be made to look like the function

$$-x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_n^2$$

and the above is its gradient. □

Remark 27.2. What do I mean by a gradient? Well, if $M \subset \mathbb{R}^n$ were submanifold of $\mathbb{R}^n$, and f were a function on M, you know what the gradient is. More generally, if we have a Riemannian manifold, we have an isomorphism between TM and T^*M given by the Riemannian metric—any vector v defines a covector $g(v, -)$. Hence given a function, we take the vector field corresponding to df via the metric. This is called the gradient of f, and it satisfies the equation

$$(\text{grad}(f), u) = df(u)$$

for every vector field u.

Lemma 27.3. Fix a metric g on E. Let s be a nowhere vanishing section of a vector bundle E. Then one can choose a compatible connection ∇ on E so that $\nabla s = 0$.

Proof of Theorem. So at each p, let's choose a nest of open sets $U_p \subset V_p \subset W_p$. We let W' be the open set $M - \bigcup_p \overline{V_p}$. We can then guarantee the following:

(1) Over W, we have no control over the metric, but $\nabla Y = 0$ and $||Y|| = 1$.
(2) Over V, we have the standard Euclidean metric, with $\nabla Y = 0$ and $||Y|| = 1$.
(3) Over U, the metric is standard Euclidean, but we have no control over $||Y||$.

How do we do this? First, we choose a metric on W that is the standard Euclidean metric; but when we patch with an arbitrary metric over W', this may change the metric. But the metric only changes where W' intersects W, so it remains the standard Euclidean metric on V and on U.

Inside U, we scale Y' into $Y'/||Y'||$ along U so that outside of U, $||Y'|| = 1$. The trick as in the lemma allows us to choose a connection outside of U for which $\nabla Y' = 0$, though we don't have control over $\nabla Y'$ along U.

Now, note that the integral $\int_{W'} Pf(\Omega_\nabla) = 0$. Why? Well, we can extend Y to an orthonormal frame locally, and the fact that $\nabla Y = 0$ means that the connection matrix α has zeroes along the first column. Since α is skew-symmetric, it also has zeroes along the first row. By the structure equation

$$\Omega = d\alpha - \alpha \wedge \alpha$$

we also see that Ω has zeroes along the first row and first column. Looking at the Pfaffian formula, $Pf(\Omega)$ must also be zero, since every product $(\Omega)\sigma(1)\sigma(2)\ldots(\Omega)_{\sigma(2k-1)\sigma(2k)}$ has at least one term from either the first row or first column. So in fact $Pf(\Omega) = 0$ as a differential form on W'.

Now we integrate the differential form $Pf(\Omega_\nabla)$ over V. Since we know the integral of $Pf(\Omega_\nabla)$ over W' is zero, so $\int_{V_p} Pf(\Omega_\nabla)$ are the only contributions to $\int_M Pf(\Omega_\nabla)$.

So now we just need to prove

$$\int_{V_p} Pf(\Omega_\nabla) = (-1)^i 2\pi$$

where i is the dimension of the simplex whose barycenter equals p. With this lemma, the proof of the theorem is complete. □

Lemma 27.4.

$$\int_{V_p} Pf(\Omega_\nabla) = (-1)^i 2\pi$$

Proof. We don't have any control over Ω_∇ inside U_p, so we'll try to reduce the problem to Stokes's Theorem. Also, we'll only perform this computation for $\dim M = 2$ today.

By the Morse Theory Lemma from above, we only need to study gradients of the functions

$$-x_1^2 - x_2^2, \qquad -x_1^2 + x_2^2, \qquad x_1^2 + x_2^2.$$

By popular vote in class, we decided to study this second function. Then its gradient, normalized, is given by

$$Y = \frac{-x_1\partial_1 + x_2\partial 2}{\sqrt{x_1^2 + x_2^2}}.$$

We don't know what it looks like inside U_p, but this is what it is inside V_p, where the Riemannian metric is standard. Well, let's find a connection such that $\nabla Y = 0$. Writing

$$\nabla\partial_1 = \alpha_{12}\partial_2, \qquad \nabla\partial_2 = \alpha_{21}\partial_1,$$

and setting

$$r = \sqrt{x_1^2 + x_2^2},$$

we have that

$$\nabla(Y) = d(\frac{-x_1}{r})\partial_1 + (\frac{-x_1}{r})\alpha_{12}\partial_2 + d(\frac{x_2}{r})\partial_2 + (\frac{x_2}{r})\alpha_{21}\partial_1$$

Looking at the ∂_1 term, for instance, we conclude that

$$\frac{-x_2}{r}\alpha_{21} = \frac{-r + x_1^2 r^{-1}}{r^2} dx_1 + \frac{x_1 x_2 r^{-1}}{r^2} dx_2$$

so that we see

$$\alpha_{21} = \frac{-x_2}{r^2} dx_1 + \frac{x_1}{r^2} dx_2$$

which is otherwise known as $d\theta$. Hence $\alpha_{12} = -d\theta$. And by the structure equation, we conclude

$$\Omega_{12} = d\alpha_{12}.$$

But the Pfaffian of a skew-symmetric matrix is just its 12 entry, so

$$Pf(\Omega_{12}) = d\alpha_{12}.$$

Hence integrate over a ball in V_p, we conclude

$$\int_{D^2} d\alpha_{12} = \int_{S^1} \alpha_{12} = \int_{S^1} -d\theta = -2\pi.$$

That is,

$$\int_{V_p} eu(\Omega_\nabla) = -1.$$

□