Given a smooth curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$, we define the arclength $L(\gamma) = \int_a^b \left\| \frac{d\gamma(t)}{dt} \right\| dt$, where

$$\|\vec{v}\|_2 = \langle \vec{v}, \vec{v} \rangle^{1/2} = \left( \sum_{i=1}^{n} v_i v_j \langle \vec{e}_i, \vec{e}_j \rangle \right)^{1/2} \qquad \vec{v} = \sum v_i \vec{e}_i$$

$\mathbb{R}^n$-inner product

$$= \left( \sum_{i=1}^{n} v_i^2 \right)^{1/2}$$

$\gamma$

On a Riemannian manifold, we have a notion of inner product using the metric $g$.

$$L(\gamma) = \int_a^b \| g(\dot{x}(\gamma(t)), \dot{x}(\gamma(t))) \| \, dt$$

$$= \int_a^b \sqrt{\sum_{i,j} g_{ij}(x(t)) \, \dot{x}^i(t) \, \dot{x}^j(t)} \, dt$$

$\gamma : [a,b] \longrightarrow M$.

$$x(t) = x(\gamma(t)) = (x^1(\gamma), \ldots x^n(\gamma))$$

$$\dot{x}(t) = \frac{dx(t)}{dt}$$

"arclength" of $\gamma$ on $M$.

Classically, $s = L(\gamma)$, so its natural to write First F.

$$\left( ds^2 = \sum g_{ij}(x(t)) \, dx^i dx^j \right) \quad \text{Form.}$$

Prop. A Riemannian manifold $(M, g)$ is a metric space, if connected. Let $\gamma : [a,b] \to M$ be a piecewise smooth curve.

Define the length of $\gamma$

$$L(\gamma) = \int_{[a,b]} \sqrt{g_{ij}(x(\gamma(t)))\, \dot{x}^i(t)\, \dot{x}^j(t)}\; dt$$

$$\dot{x}^i(t) = \frac{d}{dt} x^i(\gamma(t))$$

Define the distance fn.

$$d(p,q) = \inf \{ L(\gamma) \mid \gamma : [a,b] \to M, \; \gamma(a) = p, \; \gamma(b) = q \}.$$

pf Exercise

NTS $\quad d(p,q) \geq 0 \quad \forall\, p,q \in M$

$d(p,q) > 0$ for $p \neq q$

$d(p,q) = d(q,p) \quad \forall\, p,q \in M$

Triangle inequality

$d(p,q) \leq d(p,r) + d(r,q) \quad \forall\, p,q,r \in M.$

Define the energy of $\gamma$,

$$E(\gamma) = \frac{1}{2} \int_{[a,b]} g_{ij}(x(\gamma(t))\, \dot{x}^i(t)\, \dot{x}^j(t)\, dt$$

By Hölders inequality $\int_a^b |f \cdot 1| \le \sqrt{b-a} \left( \int_a^b |f|^2 \right)^{1/2}$

$$L(\gamma)^2 \le 2(b-a)\, E(\gamma)$$

w/ equality iff $\underbrace{g_{ij}(x(\gamma(t))\, \dot{x}^i(t)\, \dot{x}^j(t)}_{= \langle \dot{x}, \dot{x} \rangle}$ is constant.

Recall the Euler-Lagrange Eqs. of a functional

$$I(x) = \int_a^b L(t, x(t), \dot{x}(t))\, dt \quad \text{(action)}$$

are $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \quad 1 \le i \le n$. (extremize)

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Prop. The EL eqs for $E(\gamma)$ are

$$(*) \quad \ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0 \qquad (x(t) = x(\gamma(t))$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( g_{jl,k} + g_{kl,j} - g_{jk,l} \right), \quad g_{jl,k} = \frac{\partial}{\partial x^k} g_{jl}$$

(Christoffel symbol)

$$\left(L = \frac{1}{2} g_{jk}(x(t))\, \dot{x}^j \dot{x}^k\right)$$

$$\frac{d}{dt}\left(g_{ik}\dot{x}^k + g_{ji}\dot{x}^j\right) - g_{jk,i}\dot{x}^j\dot{x}^k = 0$$

$$\Rightarrow \quad g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,l}\dot{x}^l\dot{x}^k +$$

$$g_{ji,l}\dot{x}^l\dot{x}^j - g_{jk,i}\dot{x}^j\dot{x}^k = 0$$

Using $g_{ij} = g_{ji}$,

$$(**) \quad 2 g_{lm}\ddot{x}^m + \left(g_{lk,j} + g_{jl,k} - g_{jk,l}\right)\dot{x}^j\dot{x}^k = 0$$

$$\delta^i_m \quad g^{il} g_{lm}\ddot{x}^m + \frac{1}{2} g^{il}\left(g_{lk,j} + g_{jl,k} - g_{jk,l}\right)\dot{x}^j\dot{x}^k = 0$$

defn

$$\ddot{x}^i + \frac{1}{2}\Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

$\square$

$\gamma$ which satisfies this is a <u>geodesic</u>.

If $H = \frac{1}{2} g^{ij} p_i p_j$ on $T^*M|_U \simeq U \times \mathbb{R}^n$

Prop. EL $\Leftrightarrow$ Hamilton

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = g^{ij}(x)\, p_j$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2} g^{jk}{}_{,i}(x)\, p_j p_k \qquad g^{jk}{}_{,i} = \frac{\partial}{\partial x_i} g^{jk}$$

pf
$$\ddot{x}^i = g^{ij} \dot{p}_j + g^{ij}{}_{,k} \dot{x}^k p_j \qquad 1^{\underline{st}} \text{ eq.}$$

$$= g^{ij} \dot{p}_j + g^{ij}{}_{,k} \dot{x}^k g_{jl} \dot{x}^l$$

$2^{\underline{nd}}$ eq

$$\ddot{x}^i = -\frac{1}{2} \underline{g^{ij} g^{lk}{}_{,j}}\, p_l p_k + \underline{g^{ij}{}_{,k}}\, g_{jl} \dot{x}^k \dot{x}^l$$

$$= \underline{\frac{1}{2} g^{ij} g^{lm} g_{mn,j} g^{nk}}\, \overbrace{g_{lr} \dot{x}^r}^{p} \overbrace{g_{ks} \dot{x}^s}^{p}$$

$$- \underline{g^{im} g_{mn,k} g^{nj}}\, g_{jl} \dot{x}^k \dot{x}^l$$

Since $g^{ij}{}_{,l} = -g^{im} g_{mn,l} g^{nj}$ from $g^{ij} g_{jk} = \delta$

$$= \frac{1}{2} g^{ij} g_{mn,j} \dot{x}^m \dot{x}^n - g^{im} g_{mn,k} \dot{x}^k \dot{x}^n$$

$$= \frac{1}{2} g^{ij} (g_{mn,j} - g_{jn,m} - g_{jm,n}) \dot{x}^m \dot{x}^n$$

Since $g_{mn,k} \dot{x}^k \dot{x}^n = \frac{1}{2} g_{mn,k} \dot{x}^k \dot{x}^n +$

$$\frac{1}{2} g_{mk,n} \dot{x}^k \dot{x}^n$$

relabel

$$= -\Gamma^i_{mn} \dot{x}^m \dot{x}^n$$

$\boxed{\Sigma}$

$\gamma$ cogeodesic on $T^*M$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x^i} \dot{x}^i + \frac{\partial H}{\partial p^i} \dot{p}^i$$

$$= -\dot{p}_i \dot{x}^i + \dot{x}^i \dot{p}_i$$

$$= 0.$$

(Conservation of Energy.)

Define the Energy of the path $\gamma$ by

$$E(\gamma) = \frac{1}{2} \int_{[a,b]} g_{ij}(x(\gamma(t)))\, \dot{x}^i(t)\, \dot{x}^j(t)\, dt$$

By Hölder inequality, $\int_a^b |f \cdot g| \leq \left(\int_a^b |f|^2\right)^{1/2} \left(\int_a^b |g|^2\right)^{1/2}$

w/ eq. iff $f, g$ are const.

$$L(\gamma)^2 \leq \left(\int_{[a,b]} 1\, dt\right) \left(\int_{[a,b]} g_{ij}\, \dot{x}^i \dot{x}^j\, dt\right)$$

$$= 2(b-a)\, E(\gamma).$$

w/ equality iff $g_{ij} \dot{x}^i \dot{x}^j$ is constant.

Note: $\langle \dot{x}, \dot{x} \rangle = g_{ij} \dot{x}^i \dot{x}^j$, so

$$\frac{d\langle \dot{x}, \dot{x} \rangle}{dt} = g_{ij}\, \ddot{x}^i \dot{x}^j + g_{ij}\, \dot{x}^i \ddot{x}^j + g_{ij,k}\, \dot{x}^i \dot{x}^j \dot{x}^k$$

(chain rule)

$$\left(\frac{dL}{dt} = 0\right) \quad = -\left(g_{jk,l} + g_{lj,k} - g_{lk,j}\right) \dot{x}^l \dot{x}^k \dot{x}^j$$

$$+ g_{lj,k}\, \dot{x}^k \dot{x}^l \dot{x}^j$$

by $*$, $**$

$$= 0 \quad \text{since } g_{jk,l}\, \dot{x}^l \dot{x}^k \dot{x}^j = g_{lk,j}\, \dot{x}^l \dot{x}^k \dot{x}^j$$

Thm. Let $M$ be a Riemannian manifold, $p \in M$, $v \in T_pM$. Then $\exists\ \varepsilon > 0$ and exactly one geodesic $\gamma : [0, \varepsilon] \to M$ w/ $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. In particular, $\gamma$ is smooth in $p$ and $v$.

pf. $(*)$ is a 2nd order system of ODE. Picard-Lindelöf theorem on existence / uniqueness implies the claim w/ initial data given.

# Fundamental Theorem of Riemannian Geometry

Let $(M, g)$ be a Riemannian manifold (or p-RM).
Then there is a unique connection $\nabla$ which satisfies the following conditions:

($X : M \to TM$ smooth section of the tangent bundle)

1) For any (tangent) vector fields $X, Y, Z \in \Gamma(TM)$

(compatible w/ metric) $$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

where $X\langle \cdot, \cdot\rangle$ denotes the derivative of the fn. $\langle Y, Z\rangle$ along $X$. (metric tensor is preserved by parallel transport)

2) for any (tangent) vector fields $X, Y \in \Gamma(TM) = C^\infty(M, TM)$

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $[\,,\,]$ denotes the Lie Bracket.

(zero torsion)

$$\left( X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \right)$$

also written this way

pf Consider the canonical vector fields

$$\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^{n} \quad \text{and metric} \quad g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j} \right\rangle$$

locally

A connection $\nabla$ is determined by the equation

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle \quad \text{for all } i, j, k$$

$$\Updownarrow$$

$n^3$ function

Christoffel symbols

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{l} \Gamma_{ij}^{l} \frac{\partial}{\partial x_l}$$

torsion free $\Rightarrow$ $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$

Compatibility $\Rightarrow$ $\frac{\partial}{\partial x_k} g_{ij} = \left\langle \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle + \left\langle \frac{\partial}{\partial x_i}, \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j} \right\rangle$

$$\Downarrow$$

$$\left\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle = \frac{1}{2} \left( \partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik} \right)$$

unique solutions.

$$\Longrightarrow \langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \sum_l \Gamma_{ij}^{l} g_{lk}$$

(physicist notation $g_{ik,j}$)

$$\Longrightarrow \Gamma_{ij}^{l} = \frac{1}{2}\left(\sum_k \partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right) g^{kl}$$

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$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_l \Gamma_{ij}^{l} \frac{\partial}{\partial x_l}$$

Levi-civita connection

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Einstein tensor

$$G_{\alpha\beta} = \left(\delta_\alpha^\gamma \delta_\beta^\zeta - \frac{1}{2} g_{\alpha\beta} g^{\gamma\zeta}\right)\left(\Gamma^{\varepsilon}_{\gamma\zeta,\varepsilon} - \Gamma^{\varepsilon}_{\gamma\varepsilon,\zeta} + \Gamma^{\varepsilon}_{\varepsilon\sigma}\Gamma^{\sigma}_{\gamma\zeta} - \Gamma^{\varepsilon}_{\zeta\sigma}\Gamma^{\sigma}_{\varepsilon\gamma}\right)$$

( 60 terms before cancellations )

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein Field Eq.
w/ cosmological const. $\Lambda$.