Stokes Theorem

If M is an oriented, smooth manifold of dimension n and with boundary ∂M (whose orientation is induced by that of M). Let ω be a C^∞-$(n-1)$ form on M ($\Omega^{n-1}(M)$) satisfying

* if M is compact, — or
* if M is not compact, has compact support

then

$$\int_M d\omega = \int_{\partial M} j^*\omega \quad \text{where } j^*: M \hookleftarrow \partial M.$$

$$\left(\text{often simply written as } \int_{\partial M} \omega \right).$$

pf Last class or any standard text on the Analysis/Calculus on Manifolds (e.g. Spivak, etc.) Munkres

Corollary If M is an oriented, compact, smooth manifold of dimension n and has no boundary. Let $\omega \in \Omega^{n-1}(M)$ then $\int_M d\omega = 0$.

A few applications of Stokes Theorem:

Prop. (FTC of $\Omega^1(\mathbb{R}^n)$)

Let $\omega = df$ be an exact 1-form on an open subset $U \subset \mathbb{R}^n$. Let $\gamma : [a,b] \longrightarrow U$ be a path. Then

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

which is independent of path and depends only on the homotopy class $[\gamma]$.

pf. Recall for smooth maps $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$,

$\phi^* d = d \phi^*$. Hence, $\gamma^* \omega = d\, \gamma^* f$.

Let $h(t) = \gamma^* f(t) = (f \circ \gamma)(t)$. Then $\gamma^* \omega = dh$

and
$$\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega = \int_{[a,b]} dh = h(b) - h(a)$$

by the standard FTC.

$\boxtimes$

Prop. (Integration by Parts)

Let M be an oriented, smooth manifold of dimension n. Let $\zeta = dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \in \Omega^{n-1}(M)$. For $u, v \in C^\infty(M)$ and $1 \le i \le n$,

$$\int_{\partial M} uv\,\zeta = \int_M \left(\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \right) dx_i \wedge \zeta .$$

pf Define $\omega = uv\,\zeta$. Compute

$$d\omega = d(uv) \wedge \zeta + (-1)^{\deg(uv)} uv\, d\zeta$$

$$= \left(\sum_{k=1}^{n} \frac{\partial uv}{\partial x_k} dx_k \right) \wedge \zeta + uv\,(0)$$

$$= \left(\frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \right) dx_i \wedge \zeta$$

Stokes Theorem concludes the proof.

(simply connected

Corollary Suppose f is a holomorphic fn. on an open set $U \subset \mathbb{C}$ w/ boundary γ. Then

$$\oint_\gamma f(z)\,dz = 0.$$

pf. Since $f = u + iv$ is holomorphic on U, then u, v satisfy the CR eqs.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{on } U,$$

Write $dz = dx + i\,dy$, define $\omega = f\,dz$.

$$\begin{aligned} d\omega &= d\big(u\,dx - v\,dy + i(v\,dx + u\,dy)\big) \\ &= du \wedge dx - dv \wedge dy + i(dv \wedge dx + du \wedge dy) \\ &= \frac{\partial u}{\partial y} dy \wedge dx - \frac{\partial v}{\partial x} dx \wedge dy + i \frac{\partial v}{\partial y} dy \wedge dx \\ &\quad + i \frac{\partial u}{\partial x} dx \wedge dy \end{aligned}$$

$$\int_\gamma \omega = \int_U d\omega = \int_U \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + i \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right] dx \wedge dy$$

$$= 0.$$

∎

<u>Corollary</u> Suppose $D \subset \mathbb{R}^3$ is a bounded region. Suppose D inherits an orientation from the ambient space. If $P, Q, R \in C^1(D)$, then

$$\int_{\partial D} P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy =$$

$$\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

pf. Consider $\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$.

Then $d\omega = d\left(P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy \right)$

$$= \left(\sum_i \frac{\partial P}{\partial x^i} dx^i \wedge dy \wedge dz \right) + \text{similar}$$

$$= \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dz + dx$$

$$+ \frac{\partial R}{\partial z} dz \wedge dx \wedge dy$$

$$\oint_{\partial D = S} \vec{F} \cdot d\vec{A} = \int_{D = V} \vec{\nabla} \cdot \vec{F}\, d\vec{V}$$

"Flux of a vector field through a surface = volume integral of the divergence"

Corollary Let $D \subset \mathbb{R}^2$ be a bounded region w/ boundary ∂D. Suppose D inherits an orientation from the ambient space. If $P, Q \in C^1(D)$, then

$$\int_{\partial D} P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

pf Consider $\omega = P\,dx + Q\,dy$. Then

$$\begin{aligned}
d\omega &= d(P\,dx + Q\,dy) \\
&= dP \wedge dx + dQ \wedge dy \\
&= \sum_{i=1}^{2} \frac{\partial P}{\partial x^i} dx^i \wedge dx + \sum_{i=1}^{2} \frac{\partial Q}{\partial x^i} dx^i \wedge dy \\
&= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
\end{aligned}$$

Do 3D example (curl)

∎

$$\oint_{\partial D = S} \vec{F} \cdot d\vec{A} = \int_{D = V} \vec{\nabla} \times \vec{F} \cdot d\vec{V}$$ Kelvin-Stokes

Ex. Let M be a disk of radius $r > 0$ in $\mathbb{R}^2$.

Let $\omega = x\,dy$; so $d\omega = d(x\,dy) = dx \wedge dy$,

the <u>area-form</u> on $\mathbb{R}^2$. By Stokes Thm.

$$\int_{B_r^2} d\omega = \int_{S_r^1} x\,dy = \text{Area}(B_r^2)$$

Parametrize $x(\theta) = r\cos\theta$, $y(\theta) = r\sin\theta$

$$\int_{S_r^1} x\,dy = \int_0^{2\pi} r^2 \cos^2\theta\, d\theta$$

$$= \frac{r^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \pi r^2.$$

Ex. Let M be a 4-ball of radius $r > 0$ in $\mathbb{R}^4$.

Recall:

$$B^4 = \{ (x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 \leq r \}$$

Parametrize $(\sigma, \psi, \phi, \theta)$ $\quad$ σ depends on ψ

$$0 \leq \psi \leq \pi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \sigma \leq r$$

$$t = \sigma \cos \psi \qquad x = \rho \sin \phi \cos \theta$$

$$\rho = \sigma \sin \psi \qquad y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

4-content form:

$$dx \wedge dy \wedge dz \wedge dt = \frac{\partial (x, y, z, t)}{\partial (\sigma, \psi, \phi, \theta)} d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$

$$= \sigma^3 \sin^2 \psi \sin \phi \; d\sigma \wedge d\psi \wedge d\theta \wedge d\phi$$

$$\int_0^r \sigma^3 d\sigma \int_0^\pi \sin^2 \psi \, d\psi \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi = \frac{1}{2} \pi^2 r^4$$

w/ Stokes Theorem

$$\int_{B^4} dx \wedge dy \wedge dz \wedge dt = -\int_{S^3} t \, dx \wedge dy \wedge dz$$

Split into two components, upper and lower,

$$= -\int_{S^3_+} t\, dx \wedge dy \wedge dt = \int_{S^3_-} t\, dx \wedge dy \wedge dz$$

$$= 2\int_{B^3,\ x^2+y^2+z^2 \le r} \sqrt{r^2-x^2-y^2-z^2}\, dx\, dy\, dz$$

$$= 2\int_0^r \rho^2 \sqrt{r^2-\rho^2}\, d\rho \int_0^\pi \sin\phi\, d\phi \int_0^{2\pi} d\theta$$

$$= 2\left(\frac{1}{16}\pi r^4\right)(2)(2\pi)$$

$$= \frac{1}{2}\pi^2 r^4$$

Prop. In general,

$$\int \mathrm{vol}_n(B^n_r) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n \qquad \Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}$$

$$= \begin{cases} \dfrac{\pi^k}{k!} r^{2k} & n = 2k \\ \dfrac{2^{k+1}\pi^k}{(2k+1)!!} r^{2k+1} & n = 2k+1 \end{cases}$$

$(2k+1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)$

pf.

$$\mathrm{vol}_n = dx_1 \wedge \dots \wedge dx_n$$

$$= \sigma^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2}) \cdot$$

$$d\sigma \wedge d\phi_1 \wedge \dots \wedge d\phi_{n-1}$$

$$\int_{B_r^n} \mathrm{vol}_n = \left(\int_0^r \sigma^{n-1} d\sigma \right) \left(\int_0^\pi \sin^{n-2} \phi_1 \, d\phi_1 \right) \cdots$$

$$\cdots \left(\int_0^{2\pi} d\phi_{n-1} \right)$$

$$= \frac{r^n}{n} \left(2 \int_0^{\pi/2} \sin^{n-2} \phi_1 \, d\phi_1 \right) \cdots$$

$$\cdots \left(4 \int_0^{\pi/2} d\phi_{n-1} \right)$$

$$= \frac{r^n}{n} B\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) B\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right) \cdots B\left(\tfrac{2}{2}, \tfrac{1}{2}\right) \left(2 B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$$

$$= \frac{r^n}{n} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdots \frac{\Gamma\left(\frac{2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \cdot$$

$$\frac{2\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta$$

$$= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $z\Gamma(z) = \Gamma(z+1)$

$$\mathrm{Vol}_n(B_r^n) = \frac{2\pi r^2}{n}\,\mathrm{Vol}_{n-2}(B_r^{n-2})$$

$$\sim \frac{1}{\sqrt{\pi n}}\left(\frac{2\pi e}{n}\right)^{n/2} r^n \quad \text{as } n \to \infty$$

by Stirling's Approx.

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + O\left(\tfrac{1}{n}\right)\right)$$

Similarly, $B^n_{L^p,r} = \{x \in \mathbb{R}^n \mid \sum |x_i|^p \le r^p\}$

$$\mathrm{Vol}_n(B^n_{L^p,r}) = \frac{2^n\,\Gamma(\frac{1}{p}+1)^n\, r^n}{\Gamma(\frac{n}{p}+1)}$$

$$\mathrm{Vol}_n(B^n_{L^1,r}) = \frac{2^n}{n!}r^n, \qquad \mathrm{Vol}_n(B^n_{l^\infty,r}) = (2r)^n$$

Riemannian Metrics

Defn. Let M be a smooth manifold. If for each $p \in M$, there is a map $g_p : T_pM \times T_pM \to \mathbb{R}$

s.t. 1) g_p is a positive definite, inner product

2) $p \mapsto g_p$ is smooth

then $g = \{g_p, p \in M\}$ is a Riemannian metric for M.

Recall an inner product on a vector space V (over K) is a map $\rho : V \times V \to \mathbb{R}$ satisfying

1) bilinearity (form) $\quad \rho(\alpha v, w) = \alpha \rho(v, w) \qquad \forall u, v, w \in V, \ \alpha \in K$

$\rho(u+v, w) = \rho(u, w) + \rho(v, w)$

2) symmetry $\quad \rho(u, v) = \rho(v, u)^* \quad \forall u, v \in V$

3) positive-definiteness $\quad \rho(v, v) \geq 0 \quad \forall v \in V$

$\rho(v, v) = 0 \Rightarrow v = 0.$

If we relax 3) to non-degeneracy, we get a pseudo-RM.

In local coordinates, we define

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

where $\left\{\frac{\partial}{\partial x_i}\right\}$ is a basis for T_pM.

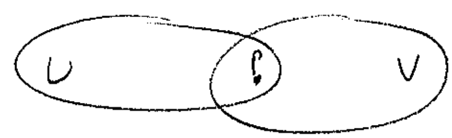
In general, given $u = \sum_i u^i \frac{\partial}{\partial x^i}\Big|_p$, $v = \sum_j v^j \frac{\partial}{\partial x^j}\Big|_p$

$$g_p(u,v) = \sum_{i,j} u^i v^j g_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

$$= \sum_{i,j} u^i v^j g_{ij}(p) \quad \text{by bilinearity}$$

Note: g is C^∞ iff $\{g_{ij}\}$ is smooth in <u>all</u> local coordinates about each p in M.

Recall: Let $(U, x), (V, y)$ be local coordinate systems about p.



$$\frac{\partial}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j} \qquad p \in U \cap V.$$

$g_{ij}(p)$ is on (U, x)

Thus,

$$g_{ij}(p) = \sum_{l,k} \frac{\partial y_k}{\partial x_i}(p) \frac{\partial y_l}{\partial x_j} \tilde{g}_{kl}(p)$$

So g is C^∞ iff $\{g_{ij}\}$ is smooth in <u>one</u> local coordinate about each point p in M.

We can use the metric to <u>define</u> an inner product on <u>covectors</u>

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

$$g^{ij}(p) := g_p(dx^i, dx^j)$$

We will show that this is little more than matrix inversion.

By definition, $g_{ij} = g_{ji}$, so it is useful to write $\bar{g}_{ij} = 2g_{ij}$ for $i<j$ and $\bar{g}_{ii} = g_{ii}$,

$$g = ds^2 = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j$$

$$= \sum_{i \le j} \bar{g}_{ij}\, dx^i dx^j. \qquad \left(dx^2 = dx\, dx \right)$$

Ex.

$$g_{ij} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \text{ on } \mathbb{R}^n. \qquad \text{standard metric}$$

$$g_{ij} = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \text{ on } \mathbb{R}^{1,n-1} \qquad \text{Lorentz metric (Minkowski space)}$$

Null-vectors: $\exists v \neq 0$ s.t. $g(v,v) = 0$

g Does not always need to be constant matrix for $\mathbb{R}^n$.

$\mathbb{R}^2$ in polar $\quad x = r\cos\theta,\ y = r\sin\theta$

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

So $g = dx^2 + dy^2$

$$= \left(\cos^2\theta\, dr^2 + r^2 \sin^2\theta\, d\theta^2 - 2r\sin\theta\cos\theta\, dr\, d\theta\right)$$

$$+ \left(\sin^2\theta\, dr^2 + r^2\cos^2\theta\, d\theta^2 + 2r\sin\theta\cos\theta\, dr\, d\theta\right)$$

$$= dr^2 + r^2 d\theta^2$$

$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ not constant!

S^2 in $\mathbb{R}^3$

$f(\phi, \theta) = (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$

$g = \begin{pmatrix} 1 & 0 \\ 0 & (\sin\phi)^2 \end{pmatrix}$ not constant!

H^2 in $\mathbb{R}^3$ $f(\phi, \theta) = (\cos\theta \sinh\phi, \sin\theta \sinh\phi, \cosh\phi,$

$g = \begin{pmatrix} 1 & 0 \\ 0 & (\sinh\phi)^2 \end{pmatrix}$ not constant!

Suppose $\{e_i\}$ is a basis of T_pM and $\{\eta^i\}$ is a basis for T^*_pM defined by $\eta^i(e_j) = \delta^i_j$.

The metric provides a map $g_p : T_pM \to T_p^*M$ by sending $e_i \longmapsto g_p(e_i, \cdot)$, so we get a covector at p.

linear functional

$$\frac{\partial}{\partial x_i} = e_i \longmapsto \sum_{j=1}^{n} g_p(\overset{\frac{\partial}{\partial x_i}}{e_i}, \overset{\frac{\partial}{\partial x_j}}{e_j})\overset{dx^j}{\eta^j} = \sum_{j=1}^{n} \overset{\delta_{ij}}{g_{ij}(p)}\, \overset{dx^j}{\eta^j} = dx^i$$

$$\sum_{i=1}^{n} v^i e_i \longmapsto \sum_{i=1}^{n} v^i \left(\sum_{j=1}^{n} g_{ij}(p)\, \eta^j \right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v^i g_{ij}(p) \right) \eta^j$$

g_{ij} is a <u>transformation</u> <u>matrix</u>.

$g^{ij}(p) = g_p(\eta^i, \eta^j)$ on the dual space

$$\eta^i \longmapsto \sum g^{ij}(p)\, e_j \qquad \sum \lambda_i \eta^i = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_i g^{ij}(p) \right)$$

However, these must be inverses, so

$$\sum_{i=1}^{n} v^i e_i = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} v^i \left(\underbrace{\sum_{j=1}^{n} g_{ij}(p)\, g^{jk}(p)}_{\delta_i^k}\right)\right) e_k$$

Thus, g^{ij} is the inverse of g_{ij}.

$$\left.\begin{aligned} g_{ij}(p) &= g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\ g^{ij}(p) &= g_p\left(dx^i, dx^j\right) \end{aligned}\right\} \text{local coordinates}$$

$$\left(g_{ij}\, g^{jk} = \delta_i^k\right) \leftarrow \text{contraction}$$

Corollary g defines an <u>isomorphism</u> from TM to T^*M.

$$\zeta \longmapsto g_p(\cdot, \zeta)$$

$$\zeta \longmapsto \sum \zeta_i\, g^{ij} = \zeta_j \;)$$

A metric can be viewed as tensor field of type $(0,2)$ taking $u_p, v_p \in T_pM$ to $g_p(u_p, v_p)$, a number.

Not just a bilinear form, but an <u>inner product</u>; which allows us to define length and angle for tangent vectors; $\|u\| = \sqrt{g_p(u,u)}$, $\|u\|\|v\|\cos\theta = g_p(u,v)$

(It is not a metric in the standard sense)
but RM is a metric space in a related way.

Recall p-times contravariant and q-times covariant tensor on M is a section of

$$\underbrace{TM \otimes \cdots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q}$$

Prop. A Riemannian metric on a smooth manifold M is a 2-times covariant tensor on M.

$\Gamma(T^*M \otimes T^*M)$

in local coordinates

$$g = \sum_{i,j} g_{ij}(x)\, dx^i \otimes dx^j$$ (First fundamental Form)

A canonical means of translating tensor fields between two smooth manifolds is by <u>pulling back</u> differential forms (contravariant vector fields) using the derivative to <u>push forward</u> vectors.

$f: M \longrightarrow N$ smooth map

$$\begin{tikzcd} TM \arrow[r, "f_*"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

(N, g) is a Riemannian manifold

We want to define $f^* g$ on M, i.e., an inner product $(f^*g)_p$ on $T_p M$.

For $v, w \in T_p M$, push forward $f_* v$, $f_* w$ in $T_{f(p)} N$, and define $(f^*g)_p (v, w) := g_{N, f(p)} (f_* v, f_* w)$

To check that $p \longmapsto (f^* g)_p$ is smooth, consider (U, x) about p, (V, y) about $f(q)$

$$f_* \frac{\partial}{\partial x^i} = \sum_{k=1}^{n} \frac{\partial (y^k \circ f)}{\partial x^i} \frac{\partial}{\partial y^k}$$

$$(f^*g)_p \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = g_{N, f(q)} \left(f_* \frac{\partial}{\partial x^i}, f_* \frac{\partial}{\partial x^j} \right)$$

$$= \sum_{k,l=1}^{n} \frac{\partial(y^k \circ f)}{\partial x^i} \frac{\partial(y^l \circ f)}{\partial x^j} \, g_{N,f(p)}\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right)$$

Since $g_{N,f(p)}$ is smooth, LHS is smooth

$\Rightarrow$ f^*g is a smooth tensor field and defines a metric on M.

If (M, g') is RM and $f^*g = g'$,

then f is an isometry.

Prop. Every smooth manifold has a Riemannian metric.

pf Let $M = \bigcup_\alpha U_\alpha$ be a covering of M by an atlas $\{(U_\alpha, \varphi_\alpha)\}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Consider the local Riemannian metric g_α in U_α, where $(g_\alpha)_{ij} = I$ in local coordinates. Define

$g = \sum_\alpha \rho_\alpha g_\alpha$ which is locally finite because the supports of ρ_α are locally finite.

Hence, g is well-defined on M and smooth, bilinear, symmetric at each point $p \in M$. Since $\rho_\alpha \geq 0$ and $\sum_\alpha \rho_\alpha = 1$, it follows that g_α is pos. def.

∎

(Rmk. $g_\alpha = \varphi^* g_{\mathbb{R}^n}$)