Lecture 24/25: Riemannian metrics

An inner product on $\mathbb{R}^n$ allows us to do the following: Given two curves intersecting at a point $x \in \mathbb{R}^n$, one can determine the angle between their tangents:

$$\cos\theta = \langle u, v\rangle / |u||v|.$$

On a general manifold, we would again like to have an inner product on each T_pM. Then if two curves intersect at a point p, one can measure the angle between them. Of course, there are other geometric invariants that arise out of this structure.

1. Riemannian and Hermitian metrics

Definition 24.2. A *Riemannian metric* on a vector bundle E is a section

$$g \in \Gamma(\mathrm{Hom}(E \otimes E, \underline{\mathbb{R}}))$$

such that g is symmetric and positive definite. A *Riemannian manifold* is a manifold together with a choice of Riemannian metric on its tangent bundle.

Chit-chat 24.3. A section of $\mathrm{Hom}(E \otimes E, \underline{\mathbb{R}})$ is a smooth choice of a linear map

$$g_p : E_p \otimes E_p \to \mathbb{R}$$

at every point p. That g is symmetric means that for every $u, v \in E_p$, we have

$$g_p(u, v) = g_p(v, u).$$

That g is positive definite means that

$$g_p(v, v) \geq 0$$

for all $v \in E_p$, and equality holds if and only if $v = 0 \in E_p$.

Chit-chat 24.4. As usual, one can try to understand g in local coordinates. If one chooses a trivializing set of linearly independent sections $\{s_i\}$, one obtains a matrix of functions

$$g_{ij} = g_p((s_i)_p, (s_j)_p).$$

By symmetry of g, this is a symmetric matrix.

Example 24.5. $T\mathbb{R}^n$ is trivial. Let $g_{ij} = \delta_{ij}$ be the constant matrix of functions, so that $g_{ij}(p) = I$ is the identity matrix for every point. Then on every fiber, g defines the usual inner product on $T_p\mathbb{R}^n \cong \mathbb{R}^n$.

Example 24.6. Let $j : M \to N$ be a smooth immersion and let h be a metric on N. Then one can define a Riemannian metric on TM by setting

$$g_p(u, v) = h_{j(p)}(Tj(u), Tj(v)).$$

We call this the *induced* or *inherited* metric. As an example, the standard sphere $j : S^2 \hookrightarrow \mathbb{R}^3$ inherits a Riemannian metric from $\mathbb{R}^3$ in this way.

Proposition 24.7. For any vector bundle E, a Riemannian metric exists.

PROOF. Partitions of unity. □

Definition 24.8. A *Hermitian* metric on a complex vector bundle E is a choice of Hermitian inner product g on each fiber E_p.

As with above, one can prove a Hermitian metric exists on any complex vector bundle E.

Definition 24.9. Two Riemannian manifolds (M, g) and (N, h) are *isometric* if there is a diffeomorphism $f : M \to N$ for which $g(u, v) = h(Tu, Tv)$.

Definition 24.10. Sections s_i are called *orthonormal* if

$$g(s_i, s_j) = \delta_{ij}.$$

If an orthonormal collection $\{s_i\}$ also spans E_p for every p, then we call $\{s_i\}$ an *orthonormal frame.*

Proposition 24.11. For any Riemannian metric on E, and for any $p \in M$, there exists a neighborhood U of p on which one can find an orthonormal frame of $E|_U$. Likewise for Hermitian metrics on a complex vector bundle.

Warning 24.12. Let g be a Riemannian metric on M. The above proposition does *not* imply that one can find a coordinate chart for M on which g looks like the identity matrix. One can find *sections* of TM for which this is true, but these sections are not induced by a coordinate chart $\mathbb{R}^n \to M$ in general. Indeed, when one can find orthonormal sections s_i such that $s_i = Tf(\partial/\partial x_i)$ for some open embedding $f : U \hookrightarrow M$, we say that the metric is *flat* on $f(U)$.

2. Levi-Civita Connection and metric connections

Given two sections s_1, s_2 of E, one can try to measure the rate of change of the function

$$g(s_1, s_2).$$

We say that a connection on E is *compatible* with g if

$$d(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2)$$

for all $s_i \in \Gamma(E)$. Note that this is an equality of 1-forms. The same equation defines the notion of compatibility of ∇ with a Hermitian connection, in the case that E is complex.

Put another way, for any pair $s_i \in \Gamma(E)$ and any vector field X, we must have

$$X(g(s_1, s_2)) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2).$$

When $E = TM$, we further say that ∇ is *torsion-free* if

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proposition 24.13. For any Riemannian metric g on E, there exists some connection that is compatible with g. If $E = TM$, there is a unique connection which is both compatible with the metric and torsion-free.

Definition 24.14. This unique connection on TM is called the *Levi-Civita connection*. For an arbitrary E, ∇ may not be unique, but is still called a *metric connection*.

3. Christoffel Symbols

Let ∇ be a connection on E. Given a local frame s_i and a local chart for the manifold, one can write

$$\nabla s_b = \Gamma^c_{ab} dx_a \otimes s_c.$$

Or, if one likes,

$$\nabla_{\partial_{x_a}} s_b = \Gamma^c_{ab} s_c.$$

In the case $E = TM$, of course, a local chart for M induces a local frame s_i on TM, and one can write

$$\nabla_{\partial_{x_a}} \partial_{x_b} = \Gamma_{ab} \partial_{x_c}.$$

The Γ^c_{ab} are called the *Christoffel symbols* for the connection ∇. If ∇ is torsion-free, we have that

$$\Gamma^c_{ab} = \Gamma^c_{ba}.$$