Friday, Oct 24, 2014

Last time:

<u>Prop'n</u> (Whitney Sum/Product Formula)

If E, F are $\mathbb{R}$ vec bundles over M,

$$p(E \oplus F) = p(E)p(F)$$

if $\mathbb{C}$-vec bundles,

$$c(E \oplus F) = c(E)c(F).$$

<u>Def'n</u> The total Pontrjagin class of a smth mfld M is defined to be

$$p(M) := p(TM) \in H^*_{dR}(M).$$

<u>Exer</u> Show $p(S^n) = 1 \in H^*_{dR}(M).$

<u>Pf</u> Let $j : S^n \hookrightarrow \mathbb{R}^{n+1}$.

Then

$$j^*(T\mathbb{R}^{n+1}) \cong TS^n \oplus \underline{\mathbb{R}}.$$

OTOH,

$$j^*(pT\mathbb{R}^{n+1}) = p(j^*(T\mathbb{R}^{n+1})).$$

$$j^*(pT\mathbb{R}^{n+1}) \overset{\text{since } T\mathbb{R}^{n+1} \text{ trivial}}{=} j^*(1) = 1$$

$$p(j^*(T\mathbb{R}^{n+1})) = p(TS^n \oplus \underline{\mathbb{R}}) \overset{\text{Whitney product}}{=} p(TS^n)\,p(\underline{\mathbb{R}}) \overset{p \cdot 1 = p}{=} p(TS^n).$$

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I want to prove that Pontrjagin numbers (to be defined) are an invariant of cobordism classes.

Integration of differential forms

Let $\alpha \in \Omega^n_{dR}(\mathbb{R}^n)$. Then $\exists!$

$$f \in C^\infty(\mathbb{R}^n)$$

s.t.

$$\alpha = f \underbrace{dx_1 \wedge \cdots \wedge dx_n}_{\text{order matters, else sign of } f \text{ could change.}}$$

If α is compactly supported,

Defn

$$\int_{\mathbb{R}^n} \alpha := \int_{\mathbb{R}^n} f \overbrace{dx_1 dx_2 \cdots dx_n}^{\text{meaningless notation from multivar. calculus}}$$

For instance,

$$\int_{\mathbb{R}^2} f \, dx_1 dx_2 = \int_{\mathbb{R}^2} f \, dx_2 dx_1$$

so do NOT interpret $dx_1 \cdots dx_n$ as a form!

Def An orientation of M is a nowhere-vanishing section

$$\eta \in \Omega_{dR}^{\dim M}(M).$$

Rmk So $\eta(p) \neq 0 \ \forall p \in M$.

Let $\phi : \mathcal{U} \longrightarrow \mathbb{R}^n$ be a chart. Then consider

$$\phi^{-1} : \phi(\mathcal{U}) \longrightarrow M.$$

If η is an orientation on M, consider its pullback. $\exists!\ f \in C^\infty(\phi(\mathcal{U}))$ s.t.

$$(\phi^{-1})^* \eta = f \, dx_1 \wedge \cdots \wedge dx_n$$

We say ϕ is positive, or that it respects η, if $f > 0$ everywhere.

Ex $\mathcal{U} = M = \mathbb{R}^2$, $\eta = dx dy = 1 dx dy$.

Then let

$$\phi^{-1} : \phi^{-1}(\mathcal{U}) \longrightarrow M$$
$$(x, y) \longmapsto (y, x).$$

Then $(\phi^{-1})^* \eta = 1 \cdot dy dx = -1 dx dy$

so this ϕ does NOT respect η.

Let α be a $(\dim M)$-form.

Let η be an orientation for M. Then

$$\int_M \alpha$$

is defined as follows:

- Choose an atlas $\mathcal{A} = \{(U_\beta, \phi_\beta)\}$ for M such that each ϕ_β^{-1} respects η.
- Choose partition of unity $\{h_\beta\}$ subordinate to $\{U_\beta\}$.

<u>Defn</u>

$$\int_M \alpha = \sum_\beta \int_{\mathbb{R}^{\dim M}} (\phi_\beta^{-1})^* (h_\beta \cdot \alpha).$$

<u>Lemma</u> This doesn't depend on choice of $\mathcal{A}$ or $\{f_\alpha\}$.

Mflds w/ Boundary

Defn $\forall\ n \in \mathbb{Z}_{\geq 0}$, let

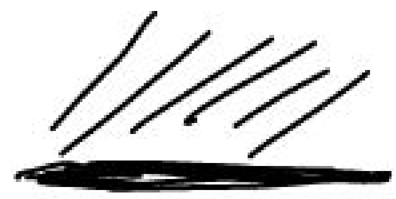
$$\mathbb{H}^n \subset \mathbb{R}^n$$

be closed upper half space.

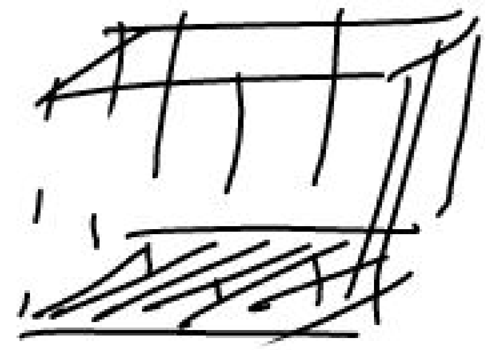
i.e.,

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \mid x_n \geq 0\}.$$

$\mathbb{H}^1$



$\mathbb{H}2$



$\mathbb{H}3$

Defn Let $\mathcal{U} \subset \mathbb{H}^m$, $V \subset \mathbb{H}^n$ open. A C^0 map

$$f : \mathcal{U} \longrightarrow V$$

is called C^∞ iff $\exists$ open sets

$$\mathcal{U} \subset \widetilde{\mathcal{U}} \subset \mathbb{R}^m$$
$$V \subset \widetilde{V} \subset \mathbb{R}^n$$

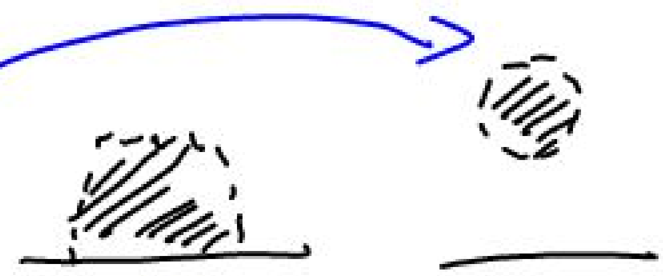
and a C^∞ fxn

$$\widetilde{f} : \widetilde{\mathcal{U}} \rightarrow \widetilde{V}$$

s.t. $f = \widetilde{f}|_{\mathcal{U}}$.

$\mathbb{H}^2$

Rmk Roughly, you should think there are two kinds of open sets in $\mathbb{H}^n$:
- Those containing some portion of $\mathbb{R}^{n-1} = \{\vec{x} \text{ s.t. } x_n = 0\}$
- Those in interior of $\mathbb{H}^n$.



Defn A fxn $\begin{array}{ccc} U & \xrightarrow{f} & V \\ \cap & & \cap \\ \mathbb{H}^n & & \mathbb{H}^n \end{array}$

is a diffeomorphism if

f is a homeomorphism, and both

f, f^{-1} are C^∞.

Defn A manifold w/ boundary

is a pair $(M, \{(U_\alpha, \phi_\alpha)\})$ where

- M is a paracompact, Hausdorff topological space
- $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is a collection where $U_\alpha \subset M$ is open, and

  $$\phi_\alpha : U_\alpha \to \mathbb{H}^n$$

  is a homeomorphism to an open subset of $\mathbb{H}^n$.

such that

- $\{U_\alpha\}$ is a cover of M, and
- $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth diffeomorphism $\forall\ \alpha, \beta$.

The boundary of M is the subset

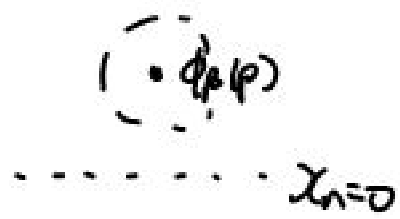
$$\partial M := \{ p \in M \mid \phi_\alpha(p) = (x_1, \ldots, x_{n-1}, 0) \text{ for some } \alpha \}.$$

Lemma $\phi_\alpha(p) = (x_1, \dots, x_{n-1}, 0)$ for some α iff

$$\phi_\alpha(p) \text{ has } x_n = 0$$

for all α.

Pf. If $\exists\, \beta$ s.t. $\phi_\beta(p)$ has $x_n > 0$,



let $V \subset \mathbb{H}^n$ be open ball s.t.

$\phi_\beta(p) \in V$.

Since $\phi_\beta \circ \phi_\alpha^{-1}$ is smth, $\exists\, \tilde{U} \subset \mathbb{R}^n$ open s.t.

- $\tilde{f} : \tilde{U} \to \mathbb{R}^n$ is C^∞
- $\tilde{f}|_U = f|_U$.

$\Rightarrow$ $\tilde{f}^{-1}(V)$ is an open set in $\mathbb{R}^n$?

$\Rightarrow$ $\tilde{f}^{-1}(V)$ could not be fully contained in $\mathbb{H}^n$, since $p \in f^{-1}(V)$ and p has x_n coordinate 0

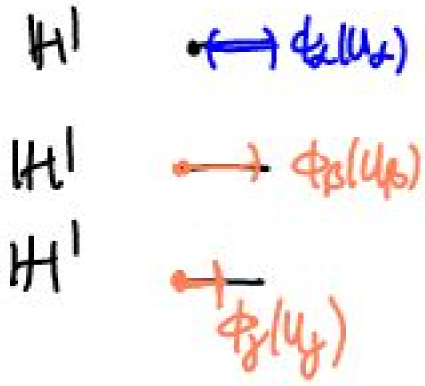
$\Rightarrow$ $f(p)$ must have had $x_n = 0$. //

Ex

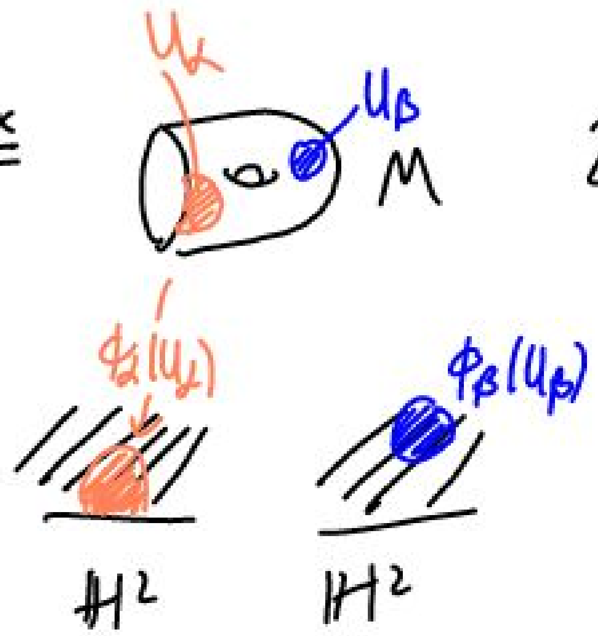
$M =$ 

1-mfld w/ ∂.

$\partial M =$ 



Ex



2-mfld w/ ∂,

$\partial M = \bigcirc$.

Rmk One can define orientation, integration for mflds w/ ∂, just as before. The important theorem is:

Thm (Stokes's Theorem).

Let M be an oriented mfld w/ ∂. Let α be a $(\dim M - 1)$ form on M, and

$$j : \partial M \hookrightarrow M$$

the inclusion of ∂M. Then

$$\int_{\partial M} j^*\alpha = \int_M d\alpha.$$