Prop'n Let E, F be smth vec bundles on M. Then

$$p(E \oplus F) = p(E) \wedge p(F)$$
$$\in H^*_{dR}(M).$$

That is,

$$p_k(E \oplus F) = \sum_{i+j=k} p_i(E) \wedge p_j(F).$$

Pf Let ∇_E, ∇_F be a connection on E, F. So

$$\nabla_E : \Gamma(E) \to \Gamma(T^*X \otimes E)$$
$$\nabla_F : \Gamma(F) \to \Gamma(T^*X \otimes F)$$

Note this defines a map of $\mathbb{R}$ vector spaces

$$\nabla_E \oplus \nabla_F : \Gamma(E) \oplus \Gamma(F) \longrightarrow \Gamma(T^*X \otimes E) \oplus \Gamma(T^*X \otimes F)$$

But note that

$$\Gamma(E) \oplus \Gamma(F) \cong \Gamma(E \oplus F).$$

After all, a section $s: M \to E \oplus F$ gives sections
$$s_E : M \longrightarrow E$$
$$s_F : M \longrightarrow F$$
by setting
$$s_E|_p : \{p\} \longrightarrow E_p \oplus F_p \to E_p$$
$$s_F|_p : \{p\} \to E_p \oplus F_p \to F_p$$

Conversely, given $(s_E, s_F) \in \Gamma(E) \oplus \Gamma(F)$, we get a section $s: M \to E \oplus F$ by $s(p) = (s_E(p), s_F(p))$.

Hence

$$\Gamma(T^*X \otimes E) \oplus \Gamma(T^*X \otimes F)$$
$$\cong$$
$$\Gamma(T^*X \otimes E \oplus T^*X \otimes F)$$
$$\cong$$
$$\Gamma(T^*X \otimes (E \oplus F))$$

So $\nabla_E \oplus \nabla_F$ is a connection on $E \oplus F$.

What is its curvature?

$$\Gamma(E \oplus F) \xrightarrow{D \circ (\nabla_E \oplus \nabla_F)} \Gamma(\Lambda^2 T^*X \otimes (E \oplus F))$$

I claim that in local coordinates we have

$$\Omega_{\nabla_E \oplus \nabla_F} = \begin{pmatrix} \Omega_{\nabla_E} & 0 \\ 0 & \Omega_{\nabla_F} \end{pmatrix}$$

Well, by the definition of the connection $\nabla_E \oplus \nabla_F$, we can conclude:

If $\{s_i^E\}_{i=1}^k$ are local LI sections for E,
and $\{s_i^F\}_{i=1}^l$ " " " " for F,

$$(\nabla_E \oplus \nabla_F)(s_i^E) = \sum \alpha_{ij}^E s_j^E + \sum 0 s_j^F$$
$$(\nabla_E \oplus \nabla_F)(s_i^F) = \sum 0 s_j^E + \sum \alpha_{ij}^F s_j^F.$$

So as a $(k+l) \times (k+l)$ matrix of 1-forms,

$$\alpha^{E \oplus F} = \begin{pmatrix} \alpha^E & 0 \\ 0 & \alpha^F \end{pmatrix}.$$

Hence by structure equation,

$$\Omega_{\nabla_E \oplus \nabla_F} = d\alpha^{E\oplus F} - \alpha^{E\oplus F} \wedge \alpha^{E\oplus F}$$

in local coords

$$= \begin{pmatrix} d\alpha^E & d0 \\ d0 & d\alpha^F \end{pmatrix} - \begin{pmatrix} \alpha^E & 0 \\ 0 & \alpha^F \end{pmatrix} \wedge \begin{pmatrix} \alpha^E & 0 \\ 0 & \alpha^F \end{pmatrix}$$

$$= \begin{pmatrix} d\alpha^E & 0 \\ 0 & d\alpha^F \end{pmatrix} - \begin{pmatrix} \alpha^E \wedge \alpha^E & 0 \\ 0 & \alpha^F \wedge \alpha^F \end{pmatrix}$$

$$= \begin{pmatrix} d\alpha^E - \alpha^E \wedge \alpha^E & 0 \\ 0 & d\alpha^F - \alpha^F \wedge \alpha^F \end{pmatrix}$$

$$= \begin{pmatrix} \Omega_{\nabla_E} & 0 \\ 0 & \Omega_{\nabla_F} \end{pmatrix}.$$

So

$$\det\left(I + \frac{1}{2\pi}\Omega_{\nabla_E \oplus \nabla_F}\right)$$

$$= \det \begin{pmatrix} I + \frac{1}{2\pi}\Omega_{\nabla_E} & 0 \\ 0 & I + \frac{1}{2\pi}\Omega_{\nabla_F} \end{pmatrix}$$

$$= \det\left(I + \frac{1}{2\pi}\Omega_E\right) \wedge \det\left(I + \frac{1}{2\pi}\Omega_F\right)$$

$$= p(E) \wedge p(F).$$

If you're having trouble keeping track of degrees,

$$\det\left(I + t\frac{1}{2\pi}\Omega_{\nabla_{E\oplus F}}\right) = \det\left(I + t\frac{1}{2\pi}\Omega_{\nabla_E}\right) \wedge \det\left(I + t\frac{1}{2\pi}\Omega_{\nabla_F}\right)$$

and t^k terms come as products of t^i and t^j terms from E and F, w/ $i+j=k$. //

The same proof says

<u>Prop'n</u> Let E, F be $\mathbb{C}$ vector bundles over M.

Then

$$c(E \oplus F) = c(E) \wedge c(F) \in H^*_{dR}(M; \mathbb{C}).$$

That is,

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) \wedge c_j(F).$$

Let's do our first computation!

<u>Prop'n</u>

$$p(\underline{\mathbb{R}}) = 1 \in H^0_{dR}(M) \subset H^*_{dR}(M)$$

$$c(\underline{\mathbb{C}}) = 1 \in H^0_{dR}(M; \mathbb{C}) \subset H^*_{dR}(M; \mathbb{C}).$$

i.e., the Pontryagin class of the trivial line bundle is trivial

<u>Pf</u> Let $\underline{\mathbb{R}}$ be given the deRham differential as a connection: $\Gamma(\underline{\mathbb{R}}) = C^\infty(M) \xrightarrow{d_{dR}} \Omega^1(M) = \Gamma(T^*X \otimes \underline{\mathbb{R}})$.

Then $D = d_{dR}$ by uniqueness of extending by Leibniz rule. So $D \circ \nabla = d_{dR}^2 = 0$.

$$\det\left(I + \tfrac{1}{2\pi} 0\right) = \det I = 1 \in C^\infty(M).$$

$$[1] = 1 \in H^0_{dR}(M)$$

Likewise, $d_{dR} \otimes \mathbb{C}$ defines a flat cnxn on $\underline{\mathbb{C}}$.

So $\det\left(I - \tfrac{1}{2\pi i} 0\right) = \det I = 1 \in C^\infty(M; \mathbb{C})$. //

Prop'n More generally, if $E \to M$ admits a flat cnxn, then $p(E) = 0$.

Rmk This makes sense — $p(E)$ was defined as an invariant whose main input was curvature.

Pf (Same as before).

Exer Let $M = S^n$, $4 \nmid n$. Prove
$$p(TM) = 1 \in H^*_{dR}(M).$$

Exer Let $M = S^{4k}$. Prove
$$p(TM) = 1 \in H^*_{dR}(M).$$

So Pontryagin classes don't tell us much about TS^n.