Lecture 21 part 1: Complex vector bundles and complex deRham cohomology.

Chit-chat 21.10. Last time I introduced Chern classes quite quickly. Let's go through things a little more carefully.

1. Direct sums of cochain complexes

Just like we can define direct sums of vector spaces, we can define direct sums of cochain complexes.

Definition 21.11. Let (A, d) be a cochain complex over $\mathbb{R}$. This means that we have a collection of vector spaces $A^k, k \in \mathbb{Z}$ over $\mathbb{R}$, and there is an $\mathbb{R}$-linear map $d : A^k \to A^{k+1}$ for every k, such that $d^2 = 0$.

Then we can define the *direct sum* cochain complex $A \oplus A$ as follows: We let $(A \oplus A)^k$ be the direct sum $A^k \oplus A^k$, and we define the differential to be

$$d(a_1, a_2) = (da_1, da_2).$$

Exercise 21.12. Prove

$$H^k(A \oplus A) \cong H^k(A) \oplus H^k(A).$$

Proof. $(a, b) \in \ker d$ if and only if $da = 0$ and $db = 0$. Hence a closed element (a, b) defines elements $[a], [b]$ of $H^k(A)$. And $[(a, b)] = [(a', b')] \in H^*(A \oplus A)$ if and only if $(a - a', b - b') = d(a'', b'')$ for some $a'', b'' \in A$. In other words, if and only if $[a] = [a']$ and $[b] = [b']$. Hence the map $[(a, b)] \mapsto ([a], [b])$ is a bijection. $\square$

2. $\Omega^*_{dR}(M; \mathbb{C})$ has no new information

So what do we mean by this notation? In what follows, we let $C^\infty(M; \mathbb{C})$ denote the set of all smooth maps from M to $\mathbb{C}$. Equivalently, it is the set of sections of the bundle $\underline{\mathbb{C}}$ over M.

Chit-chat 21.13 (As a real cochain complex)**.** We know how to take tensor products of vector bundles. So for every k, we can take the tensor product of $\Lambda^k T^*M$ with the trivial vector bundle $\underline{\mathbb{C}}$. The sections of this bundle are denoted $\Omega^k(M;\mathbb{C})$. Concretely, a section is a (linear combination of a) differential k-form tensored with a complex function.

But notice that as a *real* vector bundle, $\underline{\mathbb{C}} \cong \underline{\mathbb{R}}$. This is because $\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}$ as a real vector space. Hence we see that, as a real vector bundle,

$$\Lambda^* T^*M \otimes \mathbb{C} \cong \Lambda^* T^*M \oplus \Lambda^* T^*M.$$

This has an induced deRham differential. In fact, any section of $\Omega^k(M;\mathbb{C})$ can be written as a sum

$$\alpha + i\beta$$

where α and β are usual smooth k-forms. Then we have that

$$d(\alpha + i\beta) = d\alpha + id\beta.$$

In other words, this is the differential induced by thinking of $\Omega^k(M;\mathbb{C})$ as a direct sum.

Chit-chat 21.14 (As a complex cochain complex)**.** More naturally, $\Omega^k_{dR}(M;\mathbb{C})$ is a complex vector space for every k, and the maps $d^{\mathbb{C}}_{dR} : \Omega^k_{dR}(M;\mathbb{C}) \to \Omega^{k+1}_{dR}(M;\mathbb{C})$ are $\mathbb{C}$-linear maps. We see they are $\mathbb{C}$-linear because the complex numbers embed into $C^\infty(M;\mathbb{C})$ as the constant functions; hence $d(a\beta) = da \wedge \beta + a \cdot d\beta = a \cdot d\beta$ for any complex number a and any section $\beta \in \Omega^k_{dR}(M;\mathbb{C})$.

Then we let

$$H^k_{dR}(M;\mathbb{C}) := \ker(d^k)/\operatorname{image}(d^{k-1}).$$

denote the complex vector space given by the cohomology of this complex cochain complex.

Chit-chat 21.15. Of course, since $H^k_{dR}(M;\mathbb{C})$ is a complex vector space, we can ask what it is as a *real* vector space. The differential above shows that one can view $\Omega^*_{dR}(M;\mathbb{C})$ as a direct sum cochain complex (over $\mathbb{R}$) so that its cohomology is the direct sum of the usual deRham cohomology. Put in a cleaner way, $\mathbb{C}$ is a flat module over $\mathbb{R}$ (since it is a free module over $\mathbb{R}$) so tonsuring with $\mathbb{C}$ preserves all kernels and cokernels. This means:

$$H^*_{dR}(M;\mathbb{C}) \cong H^*_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

So there is no more information in $H^*_{dR}(M;\mathbb{C})$ then there is in $H^*_{dR}(M;\mathbb{R})$. (For instance, $\dim_{\mathbb{C}}$ of one is the same as $\dim_{\mathbb{R}}$ of the other.) One can also show there is no real ring structure to be gleaned over $\mathbb{C}$, either.

However, if you ever study complex forms on a *complex* manifold with a Hermitian metric, you will see that $H^*_{dR}(M;\mathbb{C})$ splits as a direct sum

$$H^k_{dR}(M;\mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(M;\mathbb{C})$$

where p and q dictate how *holomorphic* or *antiholomorphic* a differential form is. This is the subject of Hodge theory for Kahler manifolds, which we won't be able to talk about, unfortunately. Regardless, when you give your smooth manifold this complex, Kahler structure, you get more information—the cohomology groups split into finer groups!

Chit-chat 21.16. So that was a slight detour. But if you have time, you should go through all the Facts I mentioned in class last time, making sure that you can prove everything in the $C^\infty(M;\mathbb{C})$-linear case that we proved in the $C^\infty(M)$-linear case. I promise there are no new surprises. From now on we'll proceed; we have constructed both Pontrjagin and Chern classes. It might just take some thinking to convince yourself that the complex-vector-bundle arguments (to prove the existence of Chern classes and their invariance under change of complex connection) follow mutatis mutandis from the real vector bundle case.

3. Direct sums of vector bundles

Recall the two characteristic classes we've defined so far:

Definition 21.17. If $E \to M$ is a real vector bundle, we let

$$p_k(E) := (\frac{1}{2\pi})^2 k[s_{2k}(\Omega_\nabla)] \in H^{4k}_{dR}(M)$$

be the kth Pontrjagin class of E. The notation depends on a choice of connection ∇ on E, but as we know, the cohomology class doesn't.

Definition 21.18. If $E \to M$ is a *complex* vector bundle, we let

$$c_k(E) := (\frac{-1}{2\pi i})^k[s_k(\Omega_\nabla)] \in H^{2k}_{dR}(M;\mathbb{C})$$

be the kth *Chern class* of E. Here, ∇ is a complex connection on E, and Ω_∇ is the associated curvature, which is a section of $\Lambda^2 T^*M \otimes \mathrm{End}(E)$.

Chit-chat 21.19. Just as with last time, you can ask what the coefficients in front of s_k are about for Chern classes. And just as last time, I will tell you: There is again a classifying space for complex vector bundles. Hence one can define characteristic classes topologically just by pulling back cohomology classes from the classifying space. It's a theorem that these Chern classes—defined geometrically—match up exactly with the topologically defined characteristic classes.

Chit-chat 21.20. To really start doing computations, we'll need Hermitian metrics and Riemannian metrics on things. But Bobbie will speak about those next week. So for now, we'll take some things as given, and move forward.