Before we prove the invariance of $[f(\Omega)]$ under change of ∇, let's prove naturality:

Prop'n Let ∇ be a connection on $E \to M$. Then $\forall\ \phi : M' \to M$, $\exists !$ connection $\phi^*\nabla$ on ϕ^*E such that

$$\begin{array}{ccc} \Omega^0(\phi^*E) & \xrightarrow{\phi^*\nabla} & \Omega^1(\phi^*E) \\ \phi^* \uparrow & & \uparrow \phi^* \\ \Omega^0(E) & \xrightarrow{\nabla} & \Omega^1(E) \end{array}$$

commutes. Moreover, if we have

$$M'' \xrightarrow{\psi} M' \xrightarrow{\phi} M,$$

then $(\phi \circ \psi)^* \nabla = \psi^*(\phi^*\nabla)$.

Rmk If $\phi = \mathrm{id}_M$, $\phi^*\nabla = \nabla$.

Pf

$$\begin{array}{cccc}
\Omega^0(\phi^*E) & \left\{ M' \xrightarrow{\phi} M \xrightarrow{s} E \right\} & \Omega^1(\phi^*E) & \phi^*\alpha \otimes \phi^* s \\
\phi^* \uparrow & \uparrow & \uparrow & \uparrow \\
\Omega^0(E) & \left\{ M \xrightarrow{s} E \right\} & \Omega^1(E) & \alpha \otimes s
\end{array}$$

Let $s_i : M \to E$ be lin. ind.

sections, and

$$\nabla s_i := \sum \alpha_{ij} \otimes s_j .$$

We demand

$$\nabla' (s_i \circ \phi) = \sum \phi^* \alpha_{ij} \otimes (s_j \circ \phi).$$

So this defines ∇' locally. Taking

bump functions,

$$\begin{aligned}
\nabla' \Big(\sum f_\beta (s_i^\beta \circ \phi) \Big) &= \sum d f_\beta \otimes (s_i^\beta \circ \phi) \\
&\quad + \sum f_\beta \, \phi^* \alpha_{ij} \otimes (s_j^\beta \circ \phi) \\
&= \underbrace{d\Big(\sum_\beta f_\beta \Big)}_{\text{zero since } \sum f_\beta \text{ is constant}} \otimes \sum_i s_i^\beta \circ \phi \\
&\quad + \sum f_\beta \, \phi^* \alpha_{ij} \otimes (s_i^\beta \circ \phi) \\
&= \sum f_\beta \big(\phi^* \alpha_{i_1} \otimes (s_j \circ \phi) \big)
\end{aligned}$$

gives a well-defined connection

on $\phi^* E$. I'll leave the rest to you. //

Pf
$$\begin{tikzcd}
\Omega^0(\phi^*E) \arrow[r, "\nabla'"] & \Omega^1(\phi^*E) \arrow[r, "D'"] & \Omega^2(\phi^*E) \\
\Omega^0(E) \arrow[r, "\nabla"] \arrow[u, "\phi^*"] & \Omega^1(E) \arrow[r, "D"] \arrow[u, "\phi^*\otimes\phi^*"'] & \Omega^2(E) \arrow[u, "\phi^*\otimes\phi^*"']
\end{tikzcd}$$

Commutes, because

$$\begin{aligned}
D'(f^*\alpha \otimes (s_i\circ\phi)) &= d(f^*\alpha)\otimes(s_i\circ\phi) - f^*\alpha \wedge \nabla'(s_i\circ\phi) \\
&= f^*(d\alpha)\otimes(s_i\circ\phi) - f^*\alpha \wedge f^*\alpha_{ij}\otimes(s_j\circ\phi) \\
&= \phi^*\otimes\phi^*(d\alpha\otimes s_i) - \phi^*\otimes\phi^*(\alpha\wedge\alpha_{ij}\otimes s_j) \\
&= \phi^*\otimes\phi^*(D(\alpha\otimes s_i)).
\end{aligned}$$

What C^∞-linear map is

$$\Omega^0(\phi^*E)\to\Omega^2(\phi^*E)\ ?$$

In local coordinates,

$$\Omega_{ij} = d\alpha_{ij} - (\alpha\wedge\alpha)_{ij}.$$

So

$$\begin{aligned}
\Omega'_{ij} &= d\alpha'_{ij} - (\alpha'\wedge\alpha')_{ij} \\
&= f^*(d\alpha_{ij}) - f^*(\alpha\wedge\alpha)_{ij} \\
&= f^*\Omega_{ij}.
\end{aligned}$$

So

$$\Omega_{\nabla'} = f^*\Omega_\nabla$$

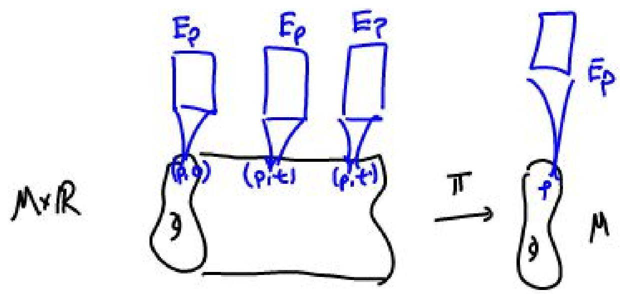
$$\Rightarrow\quad [\Omega_{\nabla'}] = f^*[\Omega_\nabla]. \quad //$$

Now let's finish the theorem.

Claim: If ∇_0, ∇_1 are connections on $E \to M$, then $\forall$ invariant polynomials f,

$$[f(\Omega_{\nabla_0})] = [f(\Omega_{\nabla_1})]$$

in $H^*_{dR}(M)$.



Pf Let $M \times \mathbb{R} \xrightarrow{\pi} M$ be the projection map. Then

$$\begin{array}{c} \pi^* E = E \times \mathbb{R}. \\ \shortparallel \\ \{(x, t, v) \mid v \in E_x\} \end{array}$$

Now consider

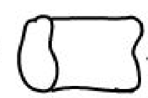
$$M \underset{j_1}{\overset{j_0}{\rightrightarrows}} M \times \mathbb{R} \xrightarrow{\pi} \underset{M.}{\overset{E}{\downarrow}}$$

Since $\pi \circ j_0 = \pi \circ j_1 = id_M$, we have

$$(\pi \circ j_t)^* E = E$$

and

$$(\pi \circ j_t)^* \nabla = \nabla.$$

$\widetilde{\nabla} = t\nabla_1 + (1-t)\nabla_0$ on  $M \times \mathbb{R}$

$$M \xrightarrow{j_0} M \times \mathbb{R} \xleftarrow{j_1} M$$

$\nabla_0 = j_0^*(\widetilde{\nabla})$ $\qquad$ $\nabla_1 = j_1^*(\widetilde{\nabla})$.

$$\begin{array}{ccc} j_0^*[f(\Omega_{\widetilde{\nabla}})] & & \\ \shortparallel & \shortparallel & \\ [f(\Omega_{\nabla_0})] & & j_1^*[f(\Omega_{\nabla_1})]. \end{array}$$

On the other hand, consider

$$\widetilde{\nabla}_0 := \pi^*\nabla_0$$

$$\widetilde{\nabla}_1 := \pi^*\nabla_1$$

connections on $\pi^*E \to M$. Then the map

$$\Gamma(\pi^*E) \to \Omega^1(\pi^*E)$$

given by

$$t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0$$

is a connection on π^*E.

Here, t is the function $M \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto t$.

For

$$\begin{aligned}(t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0)(fs) &= t\widetilde{\nabla}_1(fs) + (1-t)\widetilde{\nabla}_0(fs) \\ &= t\,df\otimes s + tf\widetilde{\nabla}_1 s + (1-t)\,df\otimes s + (1-t)f\widetilde{\nabla}_0 s \\ &= df\otimes s + f\cdot(t\widetilde{\nabla}_1 - (1-t)\widetilde{\nabla}_0)s.\end{aligned}$$

On the other hand,

$$j_0^*(t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0) = \nabla_0$$

$$j_1^*(t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0) = \nabla_1.$$

So

$$\begin{aligned}[f(\Omega_{\nabla_0})] &= j_0^*\,[f(\Omega_{t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0})] \\ &= j_1^*\,[f(\Omega_{t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0})] \\ &= [f(\Omega_{\nabla_1})]. \;//\end{aligned}$$