# Lecture 20: Pontrjagin classes

See the hand-written notes for the fact that the cohomology class $[f(\Omega_\nabla)]$ does not depend on the choice of connection. Now it's time to investigate, and give a name to, these cohomology classes.

Recall that we have defined invariant polynomials $\sigma_i$ by the equation

$$\det(I + tA) = \sum_{i=0}^{k} t^i \sigma_i.$$

Once we have Riemannian metrics and Levi-Civita connections in hand, we will be able to prove:

**Lemma 18.4.** If $i$ is odd, then for any connection,

$$[s_i(\Omega_\nabla)] = 0.$$

So we are only interested in those deRham forms obtained from even $i$.

**Definition 18.5.** We define the *$k$th Pontrajgin class* $p_k$ of a vector bundle $E$ by the equation

$$\det(1 + \frac{1}{2\pi}\Omega) := \sum_{k=0}^{\infty} p_k(E).$$

Explicitly, we have

$$p_k = (\frac{1}{2\pi})^{2k}[\sigma_{2k}(\Omega_\nabla)] \in H^{4k}_{dR}(M).$$

**Remark 18.6** (What's with the $2\pi$?)**.** You might wonder what the $2\pi$ is for.

There is always a point of tension here—do you learn algebraic topology or differential geometry first? Well, using just algebraic topology (i.e., any topological space, and not necessary a manifold) one can define the notion of *singular cohomology*. The upshot is that for any topological space $X$, one gets a sequence of abelian groups $H^0(X,\mathbb{Z}), H^1(X,\mathbb{Z})\ldots$ just like one gets a sequence of $\mathbb{R}$-vector spaces $H^*_{dR}(M)$ for any manifold $M$.

Moreover, in topology, one can construct a *classifying space $BG$*–it's some wonderful, big space with a wonderful vector bundle on it so that every bundle on $X$ is given by pulling back this wonderful vector bundle on $BG$. It's a beautiful idea—the data of a vector bundle on $X$ can be replaced by a map to $BG$.

Well, if you're pulling back bundles from $BG$, you may as well pull back certain cohomology classes of $BG$ along with it. These are the characteristic cohomology classes associated to topological vector bundles.

So we have two parallel stories about cohomology and defining characteristic classes, but the topological picture deals with abelian groups (there may even be torsion) while our picture works over $\mathbb{R}$. How to rectify this? Well, a way to get a vector space out of an abelian group is by tonsuring with $\mathbb{R}$. The deRham theorem says that there is an isomorphism

$$H^*(X;\mathbb{Z}) \otimes \mathbb{R} \cong H^*_{dR}(X).$$

Moreover, by some miracle, it turns out that the cohomology classes that we've define using *geometric methods* match exactly with the topological characteristic classes—thanks to the factors of $2\pi$ we've included.

Finally, we can repeat the construction of Pontrjagin classes—which were defined for real vector bundles—for *complex* vector bundles.

**Definition 18.7.** A *complex vector bundle of rank $k$* is the data of:

(1) A smooth manifold $E$ and a smooth map $E \to M$
(2) The structure of a complex vector space on each fiber $E_p$

such that, for every $p \in M$, there is an open set $U \subset M$ containing $p$ and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

(1) the diagram

$$\begin{tikzcd}
U \times \mathbb{C}^k \arrow[dr, "pr_1"'] & & \pi^{-1}(U) = \bigcup_{x \in U} E_x \arrow[ll, "\Phi"] \arrow[dl, "\pi"] \\
& U &
\end{tikzcd}$$

commutes—that is, so that $\pi = pr_1 \circ \Phi$,—and
(2) the map $E_x \to \{x\} \times \mathbb{C}^k$ is a $\mathbb{C}$-linear map for every $x \in U$.

A *section* of a complex vector bundle is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$.

We can define connections, the curvature 2-form, and invariant polynomials as before.

**Definition 18.8.** A connection on a complex vector bundle is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

holds for any smooth map $f : M \to \mathbb{C}$—i.e., for any smooth *complex* function on $M$.

We will have the exact same set of facts:

(1) There is a unique map $D : \Gamma(T^*M \otimes E) \to \Gamma(\Lambda^2 T^*M \otimes E)$ extending the Leibniz rule.
(2) The composition $\Omega = D \circ \nabla$ is a $C^\infty(M;\mathbb{C})$-linear map, so defines a section of $\Gamma(\mathrm{End}(E))$.
(3) For any invariant polynomial $f$, $f(\Omega)$ is closed.
(4) This cohomology class is independent of choice of connection.

So we get elements of $H^*_{dR}(M;\mathbb{C})$ out of connections on complex vector bundles.

**Definition 18.9.** We define the *kth chern class* of a complex vector bundle $E$ by the equation

$$\det(I - \frac{1}{2\pi i}\Omega) = \sum_{i=0}^{k} c_k(E).$$

These are a priori elements of $H^*_{dR}(M;\mathbb{C})$, but we will be able to prove that these are actually *real* cohomology classes once we talk about Hermitian metrics on vector bundles.