To prove invariance of

$[f(\Omega)] \in H^{2i}(\Omega)$,

recall:

<u>Prop'n</u> $\forall$ smth M,

consider

$$\begin{aligned} j_{t_0}: M &\longrightarrow M \times \mathbb{R} \\ p &\longmapsto (p, t_0) \end{aligned}$$

and

$$\begin{aligned} \pi: M \times \mathbb{R} &\longrightarrow M. \\ (p,t) &\longmapsto p \end{aligned}$$

Then

$$\pi^* \circ j_{t_0}^* = id_{H^*_{dR}(M \times \mathbb{R})}$$

and

$$j_{t_0}^* \circ \pi^* = id_{H^*_{dR}(M)}.$$

<u>Cor</u> If

$$f: M \times \mathbb{R} \longrightarrow N$$

is smooth, and

$$f_t := f(-,t): M \to N,$$

then

$$f_{t_0}^* = f_{t_1}^*: H^*_{dR}(N) \longrightarrow H^*_{dR}(M).$$

<u>Pf</u> (of Cor):

$$\begin{tikzcd} H^*_{dR}(N) \arrow[r, "f^*"] & H^*_{dR}(M \times \mathbb{R}) \arrow[r, bend left, "j_{t_1}^*"] \arrow[r, bend right, "j_{t_0}^*"'] & H^*_{dR}(M) \arrow[l, "\pi^*"'] \end{tikzcd}$$

So $f_{t_1}^* = (f \circ j_{t_1})^*$

$$= (j_{t_1})^* \circ f^* = (j_{t_0})^* \circ \pi^* \circ (j_{t_1})^* \circ f^*$$
$$= (j_{t_0})^* \circ f^* \qquad = f_{t_0}^*.$$

Pf of Prop'n

$\forall t_0, \quad \pi \circ j_{t_0} = id_M.$

Hence
$$(j_{t_0})^* \circ \pi^* = id_{H^*_{dR}(M)}.$$

We will now produce a map
$$H: \Omega^k(M \times \mathbb{R}) \longrightarrow \Omega^{k-1}(M \times \mathbb{R})$$

s.t.
$$d \circ H + H \circ d = id_{\Omega^*_{dR}(M \times \mathbb{R})} - (j_{t_0} \circ \pi)^*.$$

Then $\forall$ α closed,

$$d(H\alpha) = \alpha - (j_{t_0} \circ \pi)^* \alpha.$$

i.e., α and $(j_t \circ \pi)^* \alpha$ differ by an exact form!

Hence
$$[\alpha] = (j_{t_0} \circ \pi)^* [\alpha] \quad \forall \alpha.$$

How to define H? In local coordinates, any k-form α can be written

$$\alpha = \sum_{i_1 < \cdots < i_k} f_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$+ \sum_{j_1 < \cdots < j_{k-1}} g_J \wedge dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

Explicitly, on a point
$(x,t) \in U \times \mathbb{R}$
we have

$$\alpha(x,t) = \sum_{i_1<\cdots<i_k} f_I(x,t)\, dx_{i_1}\Big|_{(x,t)} \wedge \cdots \wedge dx_{i_k}\Big|_{(x,t)}$$

$$+ \sum_{j_1<\cdots<j_k} g_J(x,t)\, dt\Big|_{(x,t)} \wedge dx_{j_1}\Big|_{(x,t)} \wedge \cdots \wedge dx_{j_{k-1}}\Big|_{(x,t)} .$$

We define

$$H : \Omega^k(M\times\mathbb{R}) \to \Omega^{k-1}(M\times\mathbb{R})$$

in local coordinates by

$$(H\alpha)(x,t) = \sum_{j_1<\cdots<j_k} \underbrace{\left(\int_{t_0}^{t} g_J\, dt\right)}_{\substack{\text{integrate the fxn}\\ g_J \text{ over the}\\ \text{interval } [t_0,t].}} dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

Then

$$d(H\alpha)(x,t) = \sum_{j_1<\cdots<j_k} \overbrace{\frac{\partial}{\partial t}\Big|_{(x,t)} \left(\int_{t_0}^{t} g_J\, dt\right)}^{=g_J(x,t)} dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

$$+ \sum_{j_1<\cdots<j_k} \sum_i \left(\int_{t_0}^{t} \frac{\partial g_J}{\partial x_i} dt\right) dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

While

$$(d\alpha)(x,t) = \sum_{i_1<\cdots<i_k} \frac{\partial f_I}{\partial t}\Big|_{(x,t)} dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$+ \sum_{i_1<\cdots<i_k} \sum_i \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$+ \sum_{j_1\wedge\cdots\wedge j_k} \sum_i \frac{\partial g_J}{\partial x_i} dx_i \wedge dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

$$H(d\alpha)(x,t) = \sum_{i_1<\cdots<i_k} \overbrace{\left(\int_{t_0}^{t} \frac{\partial f_I}{\partial t}\Big|_{(x,t)} dt\right)}^{=f_I(x,t)-f_I(x,t_0)} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$- \sum_{j_1<\cdots<j_k} \sum_i \left(\int_{t_0}^{t} \frac{\partial g_J}{\partial x_i} dt\right) dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

Cancel!

$$\text{So } (Hd+dH)(\alpha)(x,t) = \sum f_I(x,t)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \sum g_J(x,t)\, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}}$$

$$- \underbrace{\sum f_I(x,t_0)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}}_{=\pi^* \circ j_{t_0}^*}$$

Finally, note that

$$(j_{t_0})^* \left(\sum g_J \, dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k-1}} \right) = 0$$

since

$$\text{image}\left(Tj_{t_0} \right) \subset T(M \times \mathbb{R})$$

has no $T\mathbb{R}$ component. //

Before we prove the invariance of $[f(\Omega)]$ under change of ∇, let's prove naturality:

Prop'n Let ∇ be a connection on $E \to M$. Then $\forall\ \phi : M' \to M$, $\exists!$ connection $\phi^*\nabla$ on ϕ^*E such that

$$\begin{array}{ccc} \Omega^0(\phi^*E) & \xrightarrow{\phi^*\nabla} & \Omega^1(\phi^*E) \\ \phi^* \uparrow & & \uparrow \phi^* \\ \Omega^0(E) & \xrightarrow{\nabla} & \Omega^1(E) \end{array}$$

commutes. Moreover, if we have

$$M'' \xrightarrow{\psi} M' \xrightarrow{\phi} M,$$

then $(\phi \circ \psi)^* \nabla = \psi^*(\phi^*\nabla)$.

Rmk If $\phi = \mathrm{id}_M$, $\phi^*\nabla = \nabla$.

Pf

$$\begin{array}{cccc}
\Omega^0(\phi^*E) & \left\{ \begin{array}{c} E \\ \uparrow s \\ M' \xrightarrow{\phi} M \end{array} \right\} & \Omega^1(\phi^*E) & \phi^*\alpha \otimes \phi^* s \\
\phi^* \uparrow & \uparrow & \uparrow & \uparrow \\
\Omega^0(E) & \left\{ \begin{array}{c} E \\ M \uparrow s \end{array} \right\} & \Omega^1(E) & \alpha \otimes s
\end{array}$$

Let $s_i : M \to E$ be lin. ind. sections, and

$$\nabla s_i := \sum \alpha_{ij} \otimes s_j .$$

We demand

$$\nabla' (s_i \circ \phi) = \sum \phi^* \alpha_{ij} \otimes (s_j \circ \phi).$$

So this defines ∇' locally. Taking bump functions,

$$\begin{aligned}
\nabla' \Big(\sum f_\beta (s_i^\beta \circ \phi) \Big) &= \sum df_\beta \otimes (s_i^\beta \circ \phi) \\
&\quad + \sum f_\beta \, \phi^* \alpha_{ij} \otimes (s_j^\beta \circ \phi) \\
&= \underbrace{d\Big(\sum_\beta f_\beta \Big)}_{\text{zero since } \sum f_\beta \text{ is constant}} \otimes \sum_i s_i^\beta \circ \phi \\
&\quad + \sum f_\beta \, \phi^* \alpha_{ij} \otimes (s_i^\beta \circ \phi) \\
&= \sum f_\beta \big(\phi^* \alpha_{ij} \otimes (s_j \circ \phi) \big)
\end{aligned}$$

gives a well-defined connection on ϕ^*E. I'll leave the rest to you. //

Pf
$$\begin{tikzcd} \Omega^0(\phi^*E) \arrow[r, "\nabla'"] & \Omega^1(\phi^*E) \arrow[r, "D'"] & \Omega^2(\phi^*E) \\ \Omega^0(E) \arrow[r, "\nabla"] \arrow[u, "\phi^*"] & \Omega^1(E) \arrow[r, "D"] \arrow[u, "\phi^*\otimes\phi^*"'] & \Omega^2(E) \arrow[u, "\phi^*\otimes\phi^*"'] \end{tikzcd}$$

Commutes, because

$$\begin{aligned} D'\left(f^*\alpha \otimes (s_i \circ \phi)\right) &= d(f^*\alpha) \otimes (s_i\circ\phi) - f^*\alpha \wedge \nabla'(s_i\circ\phi) \\ &= f^*(d\alpha) \otimes (s_i\circ\phi) - f^*\alpha \wedge f^*\alpha_{ij} \otimes (s_j\circ\phi) \\ &= \phi^*\otimes\phi^*(d\alpha\otimes s_i) - \phi^*\otimes\phi^*(\alpha\wedge\alpha_{ij} \otimes s_j) \\ &= \phi^*\otimes\phi^*\left(D(\alpha\otimes s_i)\right). \end{aligned}$$

What C^∞-linear map is

$$\Omega^0(\phi^*E) \to \Omega^2(\phi^*E) \ ?$$

In local coordinates,

$$\Omega_{ij} = d\alpha_{ij} - (\alpha\wedge\alpha)_{ij}.$$

So

$$\begin{aligned} \Omega'_{ij} &= d\alpha'_{ij} - (\alpha'\wedge\alpha')_{ij} \\ &= f^*(d\alpha_{ij}) - f^*(\alpha\wedge\alpha)_{ij} \\ &= f^*\Omega_{ij}. \end{aligned}$$

So

$$\Omega_{\nabla'} = f^*\Omega_\nabla$$

$$\Rightarrow \quad [\Omega_{\nabla'}] = f^*[\Omega_\nabla]. \ //$$

Now let's finish the theorem:

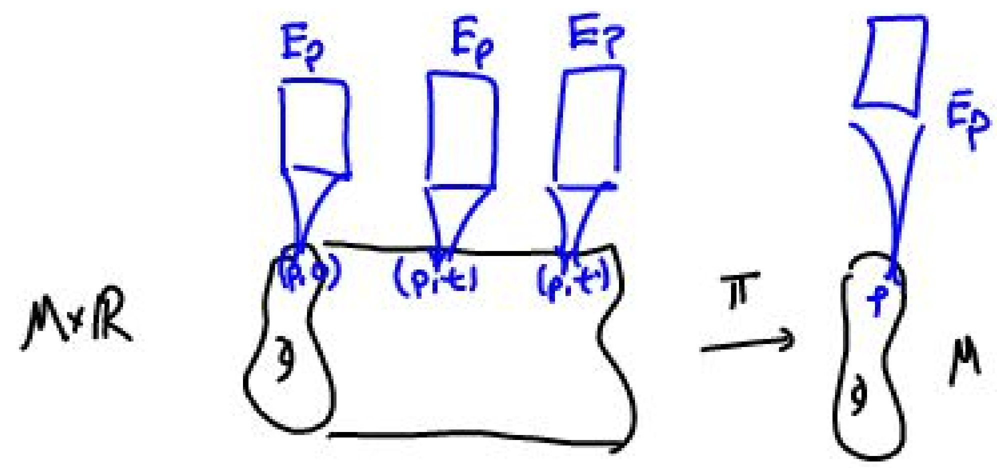
<u>Claim</u>: If ∇_0, ∇_1 are connections on $E \to M$, then $\forall$ invariant polynomials f,

$$[f(\Omega_{\nabla_0})] = [f(\Omega_{\nabla_1})]$$

in $H^*_{dR}(M)$.

Pf Let $M \times \mathbb{R} \xrightarrow{\pi} M$ be the projection map. Then

$$\underset{\{(x, t, v) \mid v \in E_x\}}{\overset{\|}{\pi^* E}} = E \times \mathbb{R}.$$



Now consider

$$M \underset{j_1}{\overset{j_0}{\rightrightarrows}} M \times \mathbb{R} \xrightarrow{\pi} \overset{\overset{E}{\downarrow}}{M}.$$

Since $\pi \circ j_0 = \pi \circ j_1 = id_M$, we have

$$(\pi \circ j_t)^* E = E$$

and

$$(\pi \circ j_t)^* \nabla = \nabla.$$

$\tilde{\nabla} = t\nabla_1 + (1-t)\nabla_0$ on $M \times \mathbb{R}$

$M \xrightarrow{j_0} M \times \mathbb{R} \xleftarrow{j_1} M$

$\nabla_0 = j_0^*(\tilde{\nabla})$

$\nabla_1 = j_1^*(\tilde{\nabla})$.

$$\underset{[f(\Omega_{\nabla_0})]}{\overset{\|}{j_0^*[f(\Omega_{\tilde{\nabla}})]}} \quad \| \quad j_1^*[f(\Omega_{\nabla_1})].$$

On the other hand, consider

$$\widetilde{\nabla}_0 := \pi^* \nabla_0$$

$$\widetilde{\nabla}_1 := \pi^* \nabla_1$$

connections on $\pi^* E \to M$. Then the map

$$\Gamma(\pi^* E) \to \Omega^1(\pi^* E)$$

given by

$$t \widetilde{\nabla}_1 + (1-t) \widetilde{\nabla}_0$$

Here, t is the function $M \times \mathbb{R} \to \mathbb{R}$
$(x,t) \mapsto t$.

is a connection on $\pi^* E$.

For

$$\begin{aligned}(t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0)(fs) &= t\widetilde{\nabla}_1(fs) + (1-t)\widetilde{\nabla}_0(fs) \\ &= t\, df \otimes s + tf \widetilde{\nabla}_1 s + (1-t) df \otimes s + (1-t) f \widetilde{\nabla}_0 s \\ &= df \otimes s + f \cdot (t\widetilde{\nabla}_1 - (1-t)\widetilde{\nabla}_0) s.\end{aligned}$$

On the other hand,

$$j_0^* (t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0) = \nabla_0$$

$$j_1^* (t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0) = \nabla_1 .$$

So

$$\begin{aligned}[f(\Omega_{\nabla_0})] &= j_0^* [f(\Omega_{t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0})] \\ &= j_1^* [f(\Omega_{t\widetilde{\nabla}_1 + (1-t)\widetilde{\nabla}_0})] \\ &= [f(\Omega_{\nabla_1})]. //\end{aligned}$$