Since each Ω_{ij} is a 2-form,

$$\Omega_{ij} \wedge \Omega_{kl} = (-1)^{|\Omega_{ij}| \cdot |\Omega_{kl}|} \Omega_{kl} \wedge \Omega_{ij}$$

$$= \Omega_{kl} \wedge \Omega_{ij}$$

i.e., "evaluating f on Ω" is well-defined, since order of multiplication doesn't matter.

Ex If $f = a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21}$,

$$f(\Omega) = \Omega_{11} + \Omega_{12} + \Omega_{11} \wedge \Omega_{22} - \Omega_{12} \wedge \Omega_{21}$$

$$\in \Omega^2(M) \oplus \Omega^4(M)$$

$$\subset \Omega^*(M).$$

If $f(BAB^{-1}) = f(A)$, $f(\Omega)$ is a globally defined form on M!

So if f is invariant,

i.e., if $f(BAB^{-1}) = f(A)$,

we see that $f(\Omega)$ is

a globally defined section

of $\Omega^*(M)$.

Moreover, if f is homogeneous

of degree k,

$$f(D \circ \nabla) \in \Omega^{2k}(M).$$

The following is the main

result we want to prove:

Thm Let ∇ be a cnxn

on E, and $\Omega = D \circ \nabla \in \Omega^2(\mathrm{End}(E))$.

Then $\forall$ homogeneous invariant

polynomials f of degree d,

(1) $f(\Omega)$ is a closed form. $\Rightarrow$ $[f(\Omega)] \in H^{2d}(M)$.

(2) If ∇, ∇' are two connections,

$$[f(\Omega)] = [f(\Omega')].$$

$\Rightarrow$ The cohomology class doesn't depend on ∇. So it's really an invariant of E!

(3) If $\phi: M' \to M$ is smooth

and $E \to M$ is a vector

bundle,

$$\phi^* [f(\Omega_\nabla)] = [f(\Omega_{\phi^*\nabla})].$$

We get an invariant for every invariant polynomial f, so we should now try to classify them all.

Given A a $k \times k$ real matrix, consider

$$\det(A + tI) = t^k + t^{k-1}\sigma_1(A) + \cdots + \sigma_k(A)$$
$$= \sum_{i=0}^{k} t^{k-i}\sigma_i(A)$$

Equivalently,

$$\det(I + tA) = \sum t^i \sigma_i.$$

$$\det(sI + tA) = \sum t^i s^{k-i} \sigma_i$$

$$s=1 \Big\downarrow \qquad \Big\downarrow t=1$$

$$\sum t^i \sigma_i \qquad \sum s^{k-i}\sigma_i$$

where σ_i is a polynomial

$$\sigma_i : M_{k\times k}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto \sigma_i(A).$$

Ex If $k=2$,

$$\det(A + tI) = \det\begin{pmatrix} a_{11}+t & a_{12} \\ a_{21} & a_{22}+t \end{pmatrix}$$
$$= t^2 + (a_{11}+a_{22})t + (a_{11}a_{22} - a_{12}a_{21})$$

so

$$\sigma_1 : A \longmapsto a_{11} + a_{22} = \operatorname{tr}(A)$$
$$\sigma_2 : A \longmapsto \det(A).$$

Note since in $M_{k\times k}(\mathbb{R}[t])$,

$$\det(BAB) = \det(A) \in \mathbb{R}[t].$$

$$\det(B(A+tI)B^{-1}) = \det(BAB^{-1} + tI) = \sum t^{k-i}\sigma_i(BAB^{-1}).$$
$$\parallel$$
$$\det(A+tI). = \sum t^{k-i}\sigma_i(A). \qquad \text{So } \sigma_i(BAB^{-1}) = \sigma_i(A)!$$

Thus we have a ring homomorphism

$$\mathbb{R}[x_1, \dots, x_k] \longrightarrow \mathrm{Inv}_k(\mathbb{R})$$

invariant polynomials $f : M_{k\times k}(\mathbb{R}) \to \mathbb{R}$.

$$x_i \longmapsto \sigma_i.$$

On the other hand, note that

$$s_i : A \mapsto \mathrm{tr}(A^i)$$

is also invariant under conjugation:

$$\begin{aligned} s_i(BAB^{-1}) &= \mathrm{tr}(BA^iB^{-1}) \\ &= \mathrm{tr}(A^i) \\ &= s_i(A). \end{aligned}$$

So we have two ring homomorphisms:

$$\mathbb{R}[\sigma_1, \dots, \sigma_k] \longrightarrow \mathrm{Inv}_k(\mathbb{R}) \longleftarrow \mathbb{R}[s_1, \dots, s_k]$$

<u>Thm</u> (Newton, essentially)

These are isomorphisms.

First note that if f is an invariant polynomial,

$$f(\mathrm{diag}(\lambda_1, \dots, \lambda_k)) = f(\mathrm{diag}(\lambda_{\sigma 1}, \dots, \lambda_{\sigma k}))$$ — just change basis by swapping order of basis elements.

for any permutation σ. So the restriction map factors through

$$\mathrm{Inv}_k(\mathbb{R}) \longrightarrow \mathbb{R}[\lambda_1, \dots, \lambda_k]^{\Sigma_k}$$

i.e., those $f \in \mathbb{R}[\lambda_1, \dots, \lambda_k]$ s.t. $f(\lambda_{\sigma 1}, \dots, \lambda_{\sigma k}) = f(\lambda_1, \dots, \lambda_k)$ $\forall \sigma \in \Sigma_k$.

$\mathbb{R}[\lambda_1, \dots, \lambda_k]^{\Sigma_k}$ = Σ_k-invariant polynomials in λ_i.

This map is a surjection — the functions σ_i are precisely the elementary symmetric functions on $\lambda_1, \dots, \lambda_k$, and these generate $\mathbb{R}[\lambda_1, \dots, \lambda_k]^{\Sigma_k}$.

To show it's an injection, we'll prove that

$$\mathrm{Inv}_k(\mathbb{C}) \to \mathbb{C}[\lambda_1 \dots \lambda_k]^{\Sigma_k}$$

is an injection. For

$$\begin{tikzcd}
\mathrm{Inv}_k(\mathbb{C}) \arrow[r] & \mathbb{C}[\lambda_1, \dots, \lambda_k]^{\Sigma_k} \\
\mathrm{Inv}_k(\mathbb{R}) \arrow[r] \arrow[u, "\text{injective}"] & \mathbb{R}[\lambda_1, \dots, \lambda_k]^{\Sigma_k} \arrow[u, "\text{injective}"']
\end{tikzcd}$$

commutes.

Why is $\mathrm{Inv}_k(\mathbb{R}) \subset \mathrm{Inv}_k(\mathbb{C})$?

Well, let $f = \sum a_I \prod x_{ij}^{p_{ij}}$ be a polynomial w/ real coeffs. Then $f \in \mathbb{R}[x_{ij}] \subset \mathbb{C}[x_{ij}]$.

Given $B = (B_{ij})$, $(B^{-1})_{ij}$ is given by rational expressions in B_{ij}. Invariance means

$$\sum a_I \prod x_{ij}^{p_{ij}} = \sum a_I \prod (B_{ik} x_{k\ell} B^{-1}_{\ell j})^{p_{ij}}$$

Before, I claimed that the polynomials

$$\sigma_i, \quad \det(I+tA) = \sum t^i \sigma_i$$

$$s_i, \quad s_i(A) = tr(A^i)$$

generated isomorphic rings.

Well, on the diagonal matrices,

$$\sigma_i(diag(\lambda_1,\dots,\lambda_k)) = \sum_{j_1<\dots<j_i} \lambda_{j_1}\cdots\lambda_{j_i}$$

and

$$s_i(diag(\lambda_1,\dots,\lambda_k)) = tr\, diag(\lambda_1^i,\dots,\lambda_k^i)$$
$$= \lambda_1^i + \dots + \lambda_k^i.$$

Then

$$\sigma_1 = \lambda_1 + \dots + \lambda_k = s_1. \qquad I+tA \quad \begin{pmatrix} 1+t\lambda_1 & 0 \\ 0 & 1+t\lambda_2 \end{pmatrix}$$

$$\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n \qquad t^2 + t(\lambda_1+\lambda_2) + t^2\lambda_1\lambda_2$$

$$= \frac{1}{2}(\sigma_1^2 - s_2).$$

ie,

$$s_2 = \sigma_1^2 - 2\sigma_2$$

For convenience, let $s_i = 0 \ \forall i \leq 0$.

Then

<u>Thm</u> (Newton's formula)
$\exists$ isomorphism

$$\mathbb{R}[s_1,\dots,s_k] \cong \mathbb{R}[\sigma_1,\dots,\sigma_k],$$

<u>Cor</u> The s_i are polynomials in σ_i, and vice versa.

Main Lemma Let

$$\Omega = D \circ \nabla \in \Omega^2(\mathrm{End}(E))$$

Then

$$s_i(\Omega) = \mathrm{tr}(\Omega^i) \in \Omega^{2i}(M)$$

is closed.

To prove this, note:

Prop'n If $\alpha, \beta \in \Omega^k(\mathrm{End}(E)), \Omega^\ell(\mathrm{End}(E))$,

then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Pf In a trivialization, let α_{ij}, β_{ij} be matrix of k, ℓ-forms, respectively.

Then

$$\begin{aligned} (d(\alpha \wedge \beta))_{ij} &= d(\alpha \wedge \beta)_{ij} \\ &= d\left(\sum_\ell \alpha_{i\ell} \wedge \beta_{\ell j}\right) \\ &= \sum_\ell d\alpha_{i\ell} \wedge \beta_{\ell j} + (-1)^{|\alpha_{i\ell}|} \alpha_{i\ell} \wedge d\beta_{\ell j} \\ &= (d\alpha \wedge \beta)_{ij} + (-1)^k (\alpha \wedge d\beta)_{ij} \end{aligned}$$

so

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

//

We'll also use

Prop'n (Bianchi identity)

Let ∇ be a cnxn,

α the matrix of 1-forms in a local triv., and

Ω the curvature 2-form.

Then

$$d\Omega = \alpha \wedge \Omega - \Omega \wedge \alpha.$$

In the sign convention where $\Omega = d\alpha + \alpha \wedge \alpha$, we'd have

$$d\Omega = \Omega \wedge \alpha - \alpha \wedge \Omega.$$

Pf. By structure equation,

$$\Omega = d\alpha - \alpha \wedge \alpha.$$

So

$$\begin{aligned} d\Omega &= d^2\alpha - d\alpha \wedge \alpha + \alpha \wedge d\alpha \\ &= -d\alpha \wedge \alpha + \alpha \wedge \alpha \wedge \alpha \\ &\qquad - \alpha \wedge \alpha \wedge \alpha + \alpha \wedge d\alpha \\ &= (-d\alpha + \alpha \wedge \alpha) \wedge \alpha \\ &\qquad - \alpha \wedge (\alpha \wedge \alpha - d\alpha) \\ &= -\Omega \wedge \alpha + \alpha \wedge \Omega. \end{aligned}$$

//

Pf (of Main Lemma)

$$d(s_i(\Omega)) = d(\mathrm{tr}(\Omega^i))$$ — Sanity check: $\mathrm{tr}(\Omega^i) \in \Omega^{2i}(M)$.

$$= d\Big(\sum_{j=1}^{k} (\Omega^i)_{jj}\Big)$$ — in local trivialization, Ω is a matrix of 2-forms, so Ω^i is a matrix of $2i$-forms. We add up its diagonal entries $(\Omega^i)_{jj}$.

$$= d\Big(\sum_{j=1}^{k} (\underbrace{\Omega\wedge\cdots\wedge\Omega}_{i \text{ times}})_{jj}\Big)$$

$$= \sum_{j=1}^{k} d\big((\Omega\wedge\cdots\wedge\Omega)_{jj}\big)$$

$$= \sum_{j=1}^{k} \big(d(\Omega\wedge\cdots\wedge\Omega)\big)_{jj}$$

$$= \mathrm{tr}(d\,\Omega^i)$$

$$= \mathrm{tr}(d\Omega\wedge\cdots\wedge\Omega + \Omega\wedge d\Omega\wedge\Omega\wedge\cdots\wedge\Omega + \Omega\wedge\cdots\wedge\Omega\wedge d\Omega).$$

How to simplify? Bianchi identity!

$$= \mathrm{tr}\big((\alpha\wedge\Omega-\Omega\wedge\alpha)\wedge\cdots\wedge\Omega + \Omega\wedge(\alpha\wedge\Omega-\Omega\wedge\alpha)\wedge\cdots\wedge\Omega + \cdots + \Omega\wedge\cdots\wedge(\alpha\wedge\Omega-\Omega\wedge\alpha)\big)$$

$$= \mathrm{tr}\big(\alpha\wedge\Omega\wedge\cdots\wedge\Omega + \underline{\Omega\wedge\alpha\wedge\Omega\wedge\cdots\wedge\Omega} + \cdots + \underline{\Omega\wedge\cdots\wedge\Omega\wedge\alpha\wedge\Omega}$$
$$- \underline{\Omega\wedge\alpha\wedge\Omega\wedge\cdots\wedge\Omega} - \underline{\Omega\wedge\Omega\wedge\alpha\wedge\Omega\wedge\cdots\wedge\Omega} \qquad - \Omega\wedge\cdots\wedge\Omega\wedge\alpha\big).$$

$$= \mathrm{tr}(\alpha\wedge\Omega^{i-1} - \Omega^{i-1}\wedge\alpha)$$ since underlined terms cancel.

$$= \mathrm{tr}(\alpha\wedge\Omega^{i-1}) - \mathrm{tr}(\Omega^{i-1}\wedge\alpha).$$

$$= 0.$$ (See next page)

//

Why does

$$\mathrm{tr}(\Omega^{i-1} \wedge \alpha) = \mathrm{tr}(\alpha \wedge \Omega^{i-1})?$$

Well,

$$\begin{aligned}
(\Omega^{i-1} \wedge \alpha)_{ab} &= \sum_{c=1}^{k} (\Omega^{i-1})_{ac} \wedge \alpha_{cb} \\
&= \sum_{c=1}^{k} (-1)^{|\alpha_{cb}| \cdot |\Omega^{i-1}_{ac}|} \alpha_{cb} \wedge (\Omega^{i-1})_{ac} \\
&= \sum_{c=1}^{k} \alpha_{cb} \cdot (\Omega^{i-1})_{ac} \quad \text{since } |\Omega^{i-1}| = |\Omega|(i-1) = 2(i-1) \text{ is even.}
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{tr}(\Omega^{i-1} \wedge \alpha) &= \sum_{a=1}^{k} (\Omega^{i-1} \wedge \alpha)_{aa} \\
&= \sum_{a=1}^{k} \sum_{b=1}^{k} \Omega^{i-1}_{ab} \wedge \alpha_{ba} \\
&= \sum_{a=1}^{k} \sum_{b=1}^{k} \alpha_{ba} \wedge \Omega^{i-1}_{ab} \\
&= \sum_{b} \sum_{a} \alpha_{ba} \wedge \Omega^{i-1}_{ab} \\
&= \sum_{b} (\alpha \wedge \Omega^{i-1})_{bb} \\
&= \mathrm{tr}(\alpha \wedge \Omega^{i-1}). \quad //
\end{aligned}$$

Rmk Same proof as $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ in $M_{k \times k}(\mathbb{R})$, just taking care of signs!

Pf of Thm (1)

Let $f \in \mathrm{Inv}_d(\mathbb{R})$.

Then

$$f = \sum_{j_1 + 2j_2 + \cdots + kj_k = d} a_{\vec{j}}\, s_1^{j_1} \cdots s_k^{j_k}$$

So

$$f(\Omega) = \sum_{\vec{j}} a_{\vec{j}}\, tr(\Omega)^{j_1} \wedge \cdots \wedge tr(\Omega^k)^{j_k}.$$

Then

$$df(\Omega) = \sum_{\vec{j}} \sum a_{\vec{j}}\, tr(\Omega)^{j_1} \wedge \cdots \left(tr(\Omega^i) \wedge \cdots \wedge \underline{\underline{d\,tr(\Omega^i)}} \wedge \cdots \wedge tr(\Omega^i)\right) \wedge \cdots \wedge \left(tr(\Omega^k)\right)^{j_k}$$

by Main Lemma.

$$= \sum_{\vec{j}} \sum a_{\vec{j}} \cdots \wedge \underline{\underline{0}} \wedge \cdots$$

$$= 0. \;//$$