Fri, Oct 10, 2014

We're headed toward Pontrjagin classes.

Prop'n

$D \circ \nabla : \Omega^0(E) \to \Omega^2(E)$ is $C^\infty(M)$-linear.

Pf $D\circ\nabla(fs) = D(df \otimes s + f\nabla s)$

$= d^2 f \otimes s - df \wedge \nabla s$

$+ df \wedge \nabla s + f D\nabla s$

$= f D\circ\nabla s.$ //

Last time, we saw that any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ is equivalent to a section of the bundle $\mathrm{Hom}(E,F)$.

Cor

$D\circ\nabla \in \Omega^2(\mathrm{End}(E))$

Pf By last time,

$D\circ\nabla \in \Gamma(\mathrm{Hom}(E, \Lambda^2 T^*M \otimes E))$

① $\cong \Gamma(E^\vee \otimes \Lambda^2 T^*M \otimes E)$

$\cong \Gamma(\Lambda^2 T^*M \otimes E^\vee \otimes E)$

② $\cong \Gamma(\Lambda^2 T^*M \otimes \mathrm{Hom}(E,E))$

just notation.

$\cong \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E))$

$\cong \Omega^2(\mathrm{End}(E))$. //

①,②: There is a natural isomorphism $\hom(V,W) \cong V^\vee \otimes W$. i.e. $\forall\, V' \to V,\ W \to W'$, the diagram

$$\begin{tikzcd} \hom(V,W) \arrow[r, phantom, "\cong"] \arrow[d] & V^\vee \otimes W \arrow[d] \\ \hom(V',W') \arrow[r, phantom, "\cong"] & V'^\vee \otimes W' \end{tikzcd}$$

commutes. So $\mathrm{Hom}(E,F) \cong E^\vee \otimes F$ as bundles!

Recall from our class on functorial constructions:

If $E \to M$ has transition functions

$g_{\alpha\beta}$ is a smooth function $U_\alpha \cap U_\beta \to GL_k(\mathbb{R})$

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k$$
$$(p, \vec{v}) \longmapsto (p, g_{\alpha\beta}(\vec{v}))$$

then $\mathrm{End}(E) \to M$ has transition functions

$$U_\alpha \cap U_\beta \times \underset{\underset{\mathbb{R}^{k^2}}{\cong}}{\mathrm{End}(\mathbb{R}^k)} \longrightarrow U_\alpha \cap U_\beta \times \mathrm{End}(\mathbb{R}^k)$$
$$(p, A) \longmapsto (p, g_{\alpha\beta} A g_{\alpha\beta}^{-1})$$

Suppose ω is a 2-form on $U_\alpha \cap U_\beta$ with values in $\mathrm{End}(\mathbb{R}^k) \cong E|_{U_\alpha \cap U_\beta}$:

$$\omega = \sum \omega_i \otimes A_i \in \Omega^2(U_\alpha \cap U_\beta) \underset{C^\infty(M)}{\otimes} \Gamma(\mathrm{End}(\underline{\mathbb{R}}^k)).$$

Changing the trivialization for E, (but not caring about ω_i in local coords), we obtain

$$\omega' = \sum \omega_i \otimes g_{\alpha\beta} A_i g_{\alpha\beta}^{-1} \in \Omega^2(U_\alpha \cap U_\beta) \underset{C^\infty(M)}{\otimes} \Gamma(\mathrm{End}(\underline{\mathbb{R}}^k)).$$

Now let $f: M_{k\times k}(\mathbb{R}) \to \mathbb{R}$ be a smooth function such that

$$f(BAB^{-1}) = f(B).$$

Then in local coordinates,

define

$$f(\omega) := \sum \omega_i \otimes f(A_i) \in \Omega^2(U_\alpha \cap U_\beta) \underset{C^\infty(M)}{\otimes} \Gamma(\underline{\mathbb{R}})$$

$$= \sum f(A_i)\, \omega_i \in \Omega^2(U_\alpha \cap U_\beta) \quad \swarrow \cong$$

In another trivialization,

$$f(\omega') = \sum \omega_i \otimes f(g_{\alpha\beta} A_i g_{\alpha\beta}^{-1})$$

$$= \sum \omega_i \otimes f(A_i)$$

$$= \sum f(A_i)$$

$$= f(\omega) \in \Omega^2(U_\alpha \cap U_\beta). \quad (*)$$

i.e., α defines a well-defined 2-form

$f(\omega)$!

<u>Rmk</u> Explicitly, $\forall\ (U_\alpha, \Phi_\alpha : E|_{U_\alpha} \cong U_\alpha \times \underline{\mathbb{R}}^k)$,

we've defined a 1-form $f_\alpha(\omega) \in \Omega^2(U_\alpha)$

that a priori depends on Φ_α. However, $(*)$ shows

it doesn't, and $\forall\ \alpha, \beta$, we have

$$f_\alpha(\omega)\Big|_{U_\alpha \cap U_\beta} = f_\beta(\omega)\Big|_{U_\alpha \cap U_\beta} \in \Omega^2(U_\alpha \cap U_\beta).$$

If you have sections $f_\alpha(\omega) \in \Omega^2(U_\alpha)$, $\{U_\alpha\}$ an open cover,

that agree on overlaps, you get a <u>global</u> section

$$f(\omega) \in \Omega^2(M)!$$

So

Prop'n Any smooth function

$$f: M_{k\times k}(\mathbb{R}) \longrightarrow \mathbb{R}$$

satisfying

$$f(BAB^{-1}) = f(A)$$

$\forall$ $A \in M_{k\times k}(\mathbb{R})$
$B \in GL_k(\mathbb{R})$

defines a map

$$\Omega^2(\mathrm{End}(E)) \to \Omega^2(M).$$

In fact, you should throw away this construction to the dustbins of history, as I simply wanted to illustrate what invariance under conjugation buys you.

So ∇ yields an element in $\Omega^2(M)$. However, this may NOT define a closed form. So we'll have issues finding invariants this way. The element $f(D\circ\nabla)$ will obviously depend on ∇; we want $[f(D\circ\nabla)] \in H^*$ to get rid of this dependence.

But we can be more creative.

Let f be a polynomial function

$$M_{k\times k}(\mathbb{R}) \longrightarrow \mathbb{R}.$$

(ie, polynomial as a function in $\mathbb{R}^{k^2}$), such that

$$f(BAB^{-1}) = f(B).$$

So

$$f(A) = \sum_{\vec{p}=(p_{11},\dots,p_{kk}) \in \mathbb{Z}^{k^2}_{\geq 0}} a_{\vec{p}} \prod_{i,j=1}^{k} A_{ij}^{p_{ij}}$$

Since the curvature form Ω_{ij} is locally a $k\times k$ matrix of 2-forms, it makes sense to evaluate f on Ω_{ij}:

$$f(\Omega) = \sum_{\vec{p}=(p_{11},\dots,p_{kk})} a_{\vec{p}} \prod_{i,j\geq 1} \Omega_{i,j}^{p_{ij}}$$

Note huge difference: Before, we applied f to Ω_{ij} ignoring the 2-form part, using only the coefficients of the matrix. Here, we're multiplying two-forms together. So this "$f(\Omega)$" is very different from the previous "$f(\Omega)$".

Since each Ω_{ij} is a 2-form,

$$\Omega_{ij} \wedge \Omega_{k\ell} = (-1)^{|\Omega_{ij}| \cdot |\Omega_{k\ell}|} \Omega_{k\ell} \wedge \Omega_{ij}$$

$$= \Omega_{k\ell} \wedge \Omega_{ij}$$

i.e., "evaluating f on Ω" is well-defined, since order of multiplication doesn't matter.

<u>Ex</u> If $f = a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21}$,

$$f(\Omega) = \Omega_{11} + \Omega_{12} + \Omega_{11} \wedge \Omega_{22} - \Omega_{12} \wedge \Omega_{21}$$

$$\in \Omega^2(M) \oplus \Omega^4(M)$$

$$\subset \Omega^*(M).$$

If $f(BAB^{-1}) = f(A)$, $f(\Omega)$ is a <u>globally</u> defined form on M!

So if f is invariant,

i.e., if $f(BAB^{-1}) = f(A)$,

we see that $f(\Omega)$ is a globally defined section of $\Omega^*(M)$.

Moreover, if f is homogeneous of degree k,

$$f(D \circ \nabla) \in \Omega^{2k}(M).$$

The following is the main result we want to prove:

Thm Let ∇ be a cnxn on E, and $\Omega = D \circ \nabla \in \Omega^2(\mathrm{End}(E))$. Then $\forall$ homogeneous invariant polynomials f of degree d,

(1) $f(\Omega)$ is a closed form. $\Rightarrow$ $[f(\Omega)] \in H^{2d}(M)$.

(2) If ∇, ∇' are two connections,
$$[f(\Omega)] = [f(\Omega')].$$
$\Rightarrow$ The cohomology class doesn't depend on ∇. So it's really an invariant of E!

We get an invariant for every invariant polynomial f, so we should now try to classify them all.

Given A a $k \times k$ real matrix, consider

$$\det(A + tI) = t^k + t^{k-1}\sigma_1(A) + \cdots + \sigma_k(A)$$
$$= \sum_{i=0}^{k} t^{k-i} \sigma_i(A)$$

where σ_i is a polynomial

$$\sigma_i : M_{k\times k}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto \sigma_i(A).$$

Ex If $k=2$,

$$\det(A+tI) = \det\begin{pmatrix} a_{11}+t & a_{12} \\ a_{21} & a_{22}+t \end{pmatrix}$$
$$= t^2 + (a_{11}+a_{22})t + (a_{11}a_{22} - a_{12}a_{21})$$

so

$$\sigma_1 : A \longmapsto a_{11} + a_{22} = tr(A)$$
$$\sigma_2 : A \longmapsto \det(A).$$

Note since in $M_{k\times k}(\mathbb{R}[t])$,

$$\det(BAB) = \det(A) \in \mathbb{R}[t].$$

$$\det(B(A+tI)B^{-1}) = \det(BAB^{-1} + tI) = \sum t^{k-i}\sigma_i(BAB^{-1}).$$

$$\det(A+tI). = \sum t^{k-i}\sigma_i(A). \quad \text{So } \sigma_i(BAB^{-1}) = \sigma_i(A)!$$

Thus we have a ring homomorphism

invariant polynomials $f: M_{k\times k}(\mathbb{R}) \to \mathbb{R}$.

$$\mathbb{R}[x_1, \ldots, x_k] \longrightarrow \mathrm{Inv}_k(\mathbb{R})$$
$$x_i \longmapsto \sigma_i.$$

On the other hand, note that

$$s_i : A \longmapsto \mathrm{tr}(A^i)$$

is also invariant under conjugation:

$$s_i(BAB^{-1}) = \mathrm{tr}(BA^iB^{-1})$$
$$= \mathrm{tr}(A^i)$$
$$= s_i(A).$$

So we have two ring homomorphisms:

$$\mathbb{R}[\sigma_1, \ldots, \sigma_k] \longrightarrow \mathrm{Inv}_k(\mathbb{R}) \longleftarrow \mathbb{R}[s_1, \ldots, s_k]$$

Thm (Newton, essentially)

These are isomorphisms.