CHAPTER 16

Curvature as a 2-form

The structure equation from last time shows that curvature can be written in local coordinates as

$$\Omega = d\alpha - \alpha \wedge \alpha.$$

The term *local coordinates* is extremely important here. The matrix of 1-forms, or 1-form of matrices, α *does not* define a global 1-form of matrices. In fact, if you change the trivialization of the bundle, α will not transform as you'd expect a 1-form to transform.

This is contrast to Ω:

Proposition 16.1. Ω defines a section of $\Omega^2(\mathrm{End}(E))$. That is, Ω is a differential 2-form taking values in the bundle of endomorphisms of E.

PROOF. We will see soon that any $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(F)$ defines a section of the bundle $E^\vee \otimes F$. So if Ω defines such a map from $\Gamma(E)$ to $\Gamma(\Lambda^2 T^*M \otimes E)$, it defines a section of

$$E^\vee \otimes \Lambda^2 T^*M \otimes E \cong \Lambda^2 T^*M \otimes E^\vee \otimes E \cong \Lambda^2 T^*M \otimes \mathrm{End}(E).$$

That's the hard part. It's easy to see that Ω defines a $C^\infty(M)$-linear map:

$$\begin{aligned} D \circ \nabla(fs) &= D(df \otimes s + f\nabla s) \\ &= d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla s + fD\nabla s \\ &= fD\nabla s. \end{aligned}$$

□

Since we know what transition functions look like for $\mathrm{End}(E)$—given transition functions for E—we have:

Corollary 16.2. In local coordinates, write $\Omega = (\Omega_{ij})$ where each Ω_{ij} is a smooth 2-form. Let $g_{\alpha\beta}$ be the transition maps for local coordinates in another coordinate system. Then the matrix of 2-forms Ω' in this new coordinate system is

$$\Omega' = g_{\alpha\beta} \Omega g_{\alpha\beta}^{-1}.$$

Remark 16.3. To be explicit, this means that Ω' is a matrix of 2-forms whose ijth entry is given by the two-forms

$$(g_{\alpha\beta} \wedge \Omega \wedge g_{\alpha\beta}^{-1})_{ij}.$$

Even more explicitly: $g_{\alpha\beta}$ is a matrix of smooth functions, as is its inverse. Wedge product of functions (i.e., 0-forms) is just multiplying by those functions, so we have ijth entry of Ω' given by the summation

$$\sum_{l,m=1}^{k} (g_{\alpha\beta})_{il} \Omega_{lm} (g_{\alpha\beta}^{-1})_{mj} = \sum_{l,m=1}^{k} (g_{\alpha\beta})_{il} (g_{\alpha\beta}^{-1})_{mj} \Omega_{lm}.$$

Chit-chat 16.4. Since Ω is a 2-form with values in $\mathrm{End}(E)$, we can think of Ω locally as a 2-form with values in a $k \times k$ matrix. (k, as usual, the rank of E.) If we change a trivialization, we saw above that the entries of the $k \times k$ matrix by a conjugation. [1]

So suppose that we have a polynomial $f : M_{k\times k}(\mathbb{R}) \to \mathbb{R}$ which is *invariant* under conjugation. That is, so that $f(A) = f(BAB^{-1})$ for any B invertible, and for any matrix A. Then any section of $\Omega^2(\mathrm{End}(E))$ defines a section of $\Omega^{2\deg f}(\underline{\mathbb{R}}) = \Omega^{2\deg f}(M)$ by post-composing with f. In other words, *every connection, together with an invariant polynomial, defines an element of* $\Omega^{2\deg f}_{dR}(M)$.

Next time, we'll investigate the space of all invariant polynomials. We'll see that

Theorem 16.5. For any invariant polynomial $f : M_{k\times k} \to \mathbb{R}$, $f(\Omega)$ is a closed form. Hence one obtains an element of $H^{2\deg f}(M)$.

1. Some practice

Exercise 16.6. Show that the matrix of 1-forms does not define a 1-form with values in $\mathrm{End}(E)$.

Exercise 16.7. Convince yourself that any invariant polynomial really turns Ω into an honest differential form.

[1] That that if we do not just change the trivialization, but also examine what the 2-form looks like in two different charts of the atlas of M, Ω will also change according to the derivatives of the transition maps for the charts.