CHAPTER 0

Addendum: The basic algebra of bundles

At first glance the following statements may seem bland, but they underly most manipulations we do with vector bundles. For instance, we will need to know when a linear map between $\Gamma(E)$ and $\Gamma(F)$ actually leads to a map of bundles $E \to F$. Likewise, we will want to know why we can write a section of $E \otimes F$ as an element of $\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

Proposition 0.1. Fix two smooth vector bundles E and F over M. The following three $C^\infty(M)$-modules are naturally isomorphic:

(1) The set of vector bundle maps $E \to F$
(2) The set of sections of the Hom bundle $\hom(E, F)$
(3) The set of $C^\infty(M)$-module homomorphisms $\hom_{C^\infty(M)}(\Gamma(E), \Gamma(F))$.

Example 0.2. Suppose we have a function $\phi : \Gamma(E) \to \gamma(F)$. If we know that for every C^∞ function $f : M \to \mathbb{R}$, $\phi(fs) = f\phi(s)$, then we can interpret ϕ as being a section of the bundle $\hom(E, F)$. We will use this, for instance, to realize that curvature is a differential 2-form with values in a vector bundle.

Proof. The equivalence between bundle maps and sections of the Hom bundle is simple, so we leave it to the reader. Given any $s \in \Gamma(E)$ and any $\phi \in \hom_{C^\infty(M)}(\Gamma(E), \Gamma(F))$, we now show that $s(p) = 0 \implies \phi(s)(p) = 0$. Then for any section s, the value of $\phi(s)$ at a point $p \in M$ depends only on the value of s at p. Hence ϕ defines a linear map $E_p \to F_p$ for every p, and the reader can check this assembles into a smooth bundle map.

In a trivializing chart U, choose a local frame e_i for E. Then $s|_U = \sum_i a_i e_i$ for some choice of smooth functions $a_i : U \to \mathbb{R}$. Take a bump function b supported inside U and for which $b(p) = 1$. Then

$$\begin{aligned} b^2\phi(s) &= \phi(b^2 s) \\ &= \phi(\sum_i (ba_i)(be_i)) \\ &= \sum_i ba_i \phi(be_i). \end{aligned}$$

Importantly, note that while a_i and e_i were only defined on U, the functions ba_i and the sections be_i are defined on all of M, hence define elements of $C^\infty(M)$ and $\Gamma(E)$. This is why we had to employ two copies of b. The result follows by noting that $s(p) = 0 \implies a_i = 0$. $\square$

Theorem 0.3. Fix E and F as before. Then there is a natural isomorphism of $C^\infty(M)$-modules

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F).$$

The above theorem requires a more substantial proof. We refer the reader to Chapter 7 of Conlon for now.