CHAPTER 15

Curvature in local coordinates

For any ∇ on a smooth vector bundle E, we've defined an operator

$$D : \Omega^k(E) \to \Omega^{k+1}(E)$$

by demanding the Leibniz rule:

$$D(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \nabla(s).$$

We've seen that $D \circ \nabla = 0$ if and only if $D^2 = 0$ (hence, if and only if $(\Omega^*(E), D)$ defines a cochain complex.)

Definition 15.1. The $\mathbb{R}$-linear map

$$D \circ \nabla : \Gamma(E) \to \Omega^2(E)$$

is called the *curvature* of ∇.

1. Connections in local coordinates

Let's see what everything looks like in local coordinates. Let $U \subset M$ be an open subset where $E|_U$ is trivial. Fix sections $s_i : U \to E|_U$, $i = 1, \ldots, k$, that are linearly independent. Then whatever ∇ does to each s_i, we define

$$\nabla(s_i) = \sum_{j=1}^{k} \alpha_{ij} \otimes s_j.$$

Here, each α_{ij} is a 1-form on U. You can consider the collection of them as a $k \times k$ matrix with values in 1-forms.

2. Matrices with values in differential forms

Taking a step back: Given any ring R, it makes sense to talk about $k \times k$ matrices with entries in R. After all, addition of matrices only requires addition of its entries, and multiplication of matrices only requires addition and multiplication of entries.

Hence it makes sense to consider the ring of $k \times k$ matrices with entries in the deRham algebra $\Omega^*(U)$ (or in $\Omega^*(M)$ for that matter).

To be completely explicit: Given two matrices

$$\alpha \in M_{k\times k}(\Omega^l(U)), \qquad \beta \in M_{k\times k}(\Omega^{l'}(U)),$$

one defines a matrix $\alpha \wedge \beta \in M_{k\times k}(\Omega^{l+l'}(U))$ by the formula

$$(\alpha \wedge \beta)_{ij} := \sum_{a=1}^{k} \alpha_{ik} \wedge \beta_{kj}.$$

One can also define a new matrix $d\alpha$ by applying the deRham differential entry by entry:

$$(d\alpha)_{ij} := d(\alpha_{ij}).$$

This will simplify many of our formulas in equations to come.

3. The structure equation

The composite function $D \circ \nabla : \Omega^0(E) \to \Omega^2(E)$ can be written using a 2-form, just as we wrote the connection using 1-forms:

$$D \circ \nabla(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j.$$

Proposition 15.2 (The structure equation)**.** If α is the matrix of 1-forms corresponding to a connection in a local frame, then

$$\Omega = d\alpha - \alpha \wedge \alpha.$$

PROOF.

$$\begin{aligned} D \circ \nabla(s_i) &= D(\sum_j \alpha_{ij} \otimes s_j) \\ &= \sum_j \left(d\alpha_{ij} \otimes s_j + (-1)^{|\alpha_{ij}|}\alpha_{ij} \wedge \nabla(s_j)\right) \\ &= \sum_j \left(d\alpha_{ij} \otimes s_j - \alpha_{ij} \wedge \sum_{l=1}^{k} \alpha_{jl} s_l\right) \\ &= \sum_{j=1}^{k} \left(d\alpha_{ij} - \sum_{l=1}^{k} \alpha_{il} \wedge \alpha_{lj}\right) \otimes s_j \end{aligned}$$

(1)

So writing things out entry by entry, we see

$$\Omega_{ij} = (d\alpha - \alpha \wedge \alpha)_{ij}.$$

□

Warning 15.3. Some books will write the structure equation as

$$\Omega = d\alpha + \alpha \wedge \alpha.$$

The reason for the sign error is in the convention for the Leibniz rule! The difference is whether one considers sections of the bundle $E \otimes \Omega^*(M)$, or $\Omega^*(M) \otimes E$. The two bundles are isomorphic, of course, but the sign convention in the Leibniz rule changes depending on the order in which you write the tensor product. Indeed, in defining D, one can demand the rules

$$D(s \otimes \alpha) = \nabla(s) \wedge \alpha + s \otimes d\alpha \qquad \text{(other books)},$$

and

$$D(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \nabla(s) \qquad \text{(our convention)}.$$

This won't cause us much trouble. But I wanted to alert you to this phenomenon.