Fri, Oct 3, 2014

Last time we showed
$$D \circ \nabla = 0 \iff D^2 = 0.$$

Defn ∇ is called <u>flat</u> if $D \circ \nabla = 0$.

We motivated this condition algebraically. to get a cochain complex. <u>BUT</u>:

① What does this mean?

② What is it good for?

We'll talk about some stories today, without 100% rigor.

<u>Geometric interpretation of flatness</u>

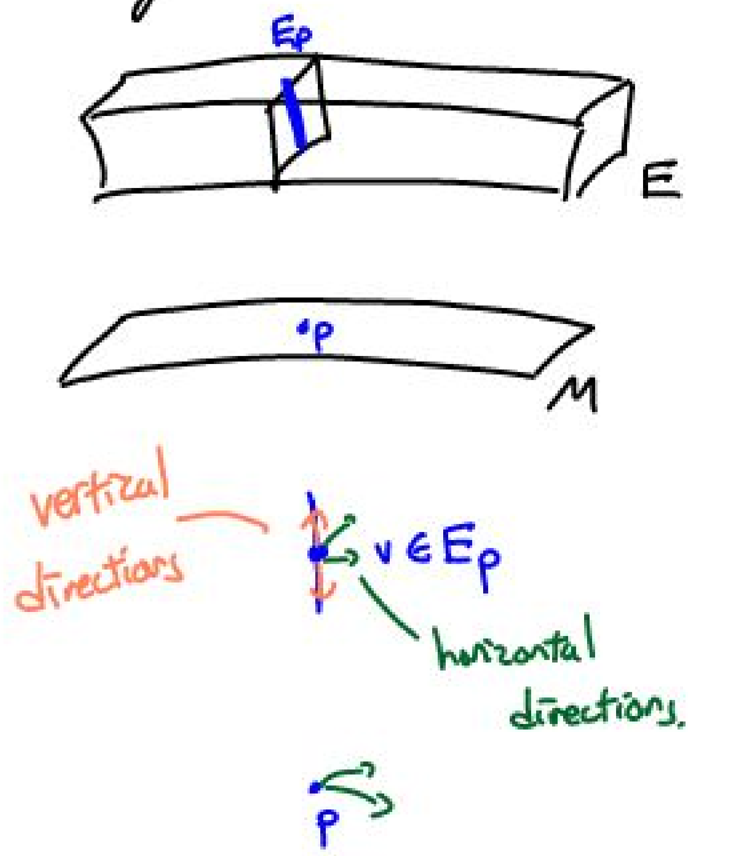
Consider the tangent bundle TE of the total space of your vector bundle.

Claim Every ∇ defines a subbundle
$$\mathcal{H}^\nabla = \mathcal{H}$$
depends on ∇, but suppress in notation.

of TE.

How would you do this?

Well, note that every $E \xrightarrow{\pi} M$ looks like it wants to have a vertical and a horizontal splitting of the tangent bundle:



In fact, the vertical directions are easy to characterize: they should be those directions with no "M component," so we should consider

$$V(E) = \mathrm{Ker}(T\pi : TE \to TM)$$

This is a subbundle of TE. We will sometimes write

$$V(E) = V.$$

So at every $v \in E$, $\pi(v) = p$,

we have a

sequence of vector spaces

$$0 \to V_v \xrightarrow[\text{injection}]{} T_vE \xrightarrow[\text{surjection.}]{} \overset{T_p(M)}{\overset{\cong}{T_vE/T_vV}} \to 0$$

As you know from ordinary linear

algebra, a splitting

$$T_vE \cong V_v \oplus T_pM$$ 1

is <u>not</u> canonical gives a sequence

as above. There are many choices

of maps

$$T_vE \longleftarrow T_pM.$$

I claim that any connection ∇

gives rise to such a splitting

— for every $v \in E$, and smoothly.

How? Two possible constructions:

(1) Consider the set of all (γ, σ) where

$$\gamma : \mathbb{R} \to M$$

is a smooth curve, and

$$\begin{tikzcd} & E \arrow[d] \\ \mathbb{R} \arrow[ur, "\sigma"] \arrow[r] & M \end{tikzcd}$$

is a smooth section <u>constant</u> with respect to ∇. Let

$$\mathcal{H} = \bigcup_{(\gamma,\sigma)} T\sigma\left(\frac{\partial}{\partial t}\right).$$

Precisely: Let $\gamma^* E$ be the pullback bundle. Then σ also defines a section of $\gamma^* E$, while ∇ induces a connection ∇' on $\gamma^* E$. We say σ is constant if $\nabla' \sigma = 0$.

These are the tangent vectors along which a section looks constant — i.e., horizontal — wrt ∇.

(2) Alternatively, $\forall$ sections

$$s : M \to E,$$

consider the linear map

$$Ts - \nabla s : TM \to TE$$

Precisely: Notice $V \cong \pi^* E$, so if $\nabla s(\vec{u}) \in E_p$, we can consider it an element of $V_{s(p)}$.

Geometrically — if $\vec{u} \in TM$, we're removing from $Ts(\vec{u}) \in T_{s(p)} E$ its non-constant part, $(\nabla s)(\vec{u}) \in E_p \cong V_{s(p)} \subset T_{s(p)} E$.

Here, we identify E_p with $T_{s(p)} E_p \subset T_{s(p)} E$ so that if $\vec{u} \in T_p M$,

$$\nabla s(\vec{u}) \in E_p \cong T_{s(p)} E_p$$

gives an element of TE_p.

Then $\mathcal{H} = \bigcup_s \operatorname{image}(Ts - \nabla s)$.

In this way (through either (1) or (2)) we obtain a subbundle

$$\mathcal{H}_\nabla = \mathcal{H} \subset TE.$$

Here's a question you may not have thought of before:

Does every subbundle

$$\mathcal{H} \subset TN$$

(for N a smth mfld) determine a family of submanifolds of N?

Put another way:

$\forall p \in N$, does there exist some submanifold

$$W \subset N$$

such that $p \in W$, and

$$TW = \mathcal{H}|_W \ ?$$

It might surprise you that the answer is <u>no</u> — there are choices of $\mathcal{H}$ s.t. a pt $p \in N$ need not admit some smth submanifold whose tangent spaces parametrize $\mathcal{H}$.

For one thing, a necessary condition should be that $\mathcal{H}$ is closed under the Lie bracket of TN. (If $W \subset N$ is a submanifold, certainly

$$TW$$

is closed under the Lie bracket.) What's amazing is that this is sufficient, too.

<u>Defn</u> A choice of subbundle $\mathcal{H}$ of the tangent bundle TN is called a <u>distribution</u>.

A distribution is called <u>integrable</u> if $\forall p \in N$, $\exists$ a submanifold $W \overset{j}{\hookrightarrow} N$ s.t. $Tj(TW) = \mathcal{H}|_{j(W)}$.

<u>Thm</u> (Frobenius theorem) $\mathcal{H} \subset TN$ is integrable iff it is closed under the Lie bracket.

So when does ∇ define a family of submanifolds for E?

<u>Thm</u> ∇ is flat iff $\mathcal{H}_\nabla$ is integrable.

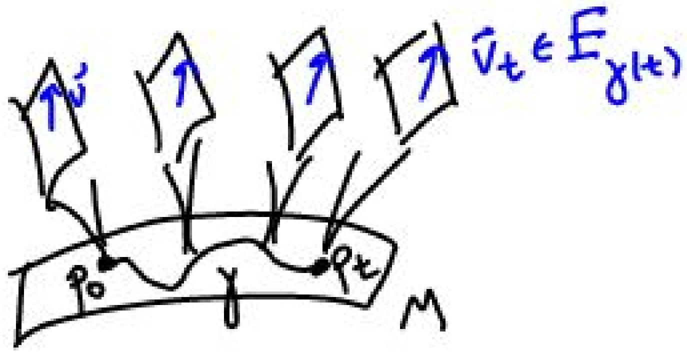
So a flat connection divides E into a bunch of submanifolds given by $\mathcal{H}$.

We also get a nice geometric interpretation of curvature: It measures to what extent the horizontal sections of ∇ are preserved by the Lie bracket.

So what are flat connections good for?

Story time: Imagine a bundle $E \to M$, and fix a point $\vec{v} \in E_{p_0}$ $\quad p_0 \in M$.

If you move along a curve $\gamma : \mathbb{R} \to M$ w/ $\gamma(0) = p_0$, on M, is there a notion of moving $\vec{v}$ from fiber to fiber "constantly"?



Well, if we have a connection on E, we have a notion of rate of change of a section. So you might expect you can define some section

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ \mathbb{R} \arrow[ur, "\sigma"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

which is "constant" along γ. This will be called <u>parallel transport of $\vec{v}$</u> along γ (wrt ∇).

In fact, doing this $\forall\ \vec{v} \in E_{p_0}$, we will get a linear isomorphism,

$$T_{\gamma,t}: E_{p_0} \longrightarrow E_{\gamma(t)}$$

for all times t.

Fact: $T_{\gamma,t}$ depends on γ.

That is, if γ and γ' are two curves w/ $\gamma(0)=\gamma'(0)$, $\gamma(t)=\gamma'(t)$, we may have that

$$T_{\gamma,t} \neq T_{\gamma',t}.$$

However:

Thm If $D \circ \nabla = 0$,

$T_{\gamma,t} = T_{\gamma',t}$ if γ and γ' are smoothly homotopic through paths respecting the endpoints $\gamma(0)=\gamma'(0)$, $\gamma(t)=\gamma'(t)$.

Cor Any flat connection on $E \to M$ induces a group homomorphism

$$\pi_1(M, p_0) \longrightarrow GL(E_{p_0}).$$

$\cong$ to $GL_k(\mathbb{R})$ if $\dim E_p = k$.

So the space of flat connections may say something about possible representations

$$\pi_1(M, p_0) \to GL_k(\mathbb{R}).$$

You can play this game w/ extra structures: For example, demanding that M be a complex holomorphic mfld, and that E be a holomorphic vector bundle.

This is the beginnings of non-Abelian Hodge theory.

⚠ Not the same thing as non-commutative Hodge theory.

Hodge theory for algebraic geom over non-commutative rings.

See: Goldman-Millson
Hitchin, "Anti-self dual ..."
Simpson

Even w/out holomorphic structures, we have the following:

Thm When M is connected, $\exists$ bijection

$$\left\{\begin{matrix}\text{Homomorphisms} \\ \pi_1(M) \to GL_n(\mathbb{R})\end{matrix}\right\} \Big/ \text{Conjugation} \cong \left\{\begin{matrix}\text{flat rank } k \\ \text{bundles } E \to M\end{matrix}\right\} \Big/ \text{Isom.}$$

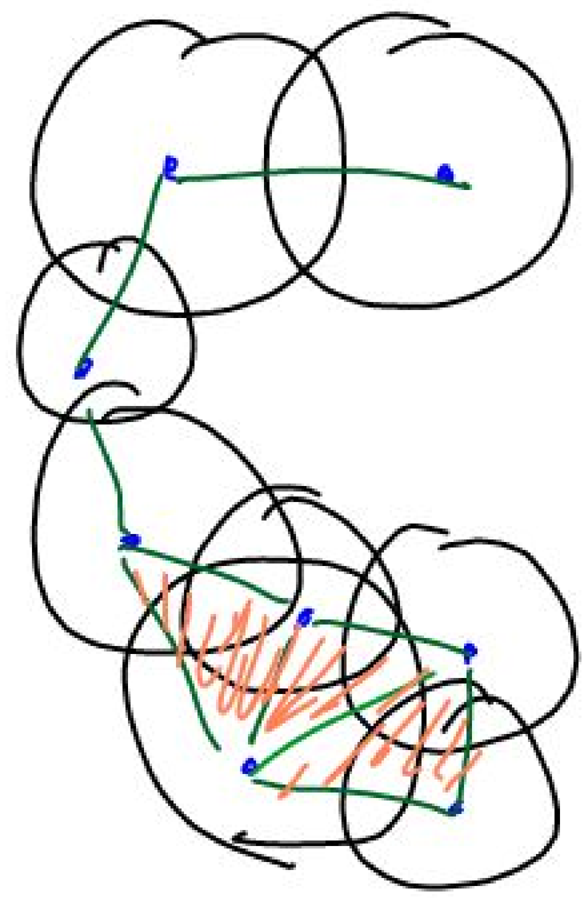
<u>Rmk</u> Of course, these sets should have more structure.

On the left, the set

$$\{\pi_1 M \longrightarrow GL_n(\mathbb{R})\}$$

is <u>algebraic</u> over $\mathbb{R}$ if M is compact. What do I mean?

If M is compact, you (a student) might expect $\pi_1 M$ to be finitely generated, w/ a finite # of relations. (Aka "finitely presented.") Why? You can model M as some shape (a simplicial set) obtained by gluing together finitely many vertices, edges, triangles, n-simplices.



(For instance, cover M by tiny open balls; construct a vertex @ the center of each, an edge between each pair of vertices whose balls intersect, and a k-simplex between each $(k+1)$-tuple of vertices whose balls intersect.) By compactness, there's a finite # of k-simplices for each k. In particular, finitely many cycles of edges (so π_1 is fin. gen.) and finitely many filling polygons (so it's fin. presented.)

Anyhow, $\pi_1 M$ is finitely presented.

Also note $GL_k(\mathbb{R})$ is algebraic, in that it's given as the set of solutions to a polynomial.

$$GL_k(\mathbb{R}) = \{ A \in \mathbb{R}^{k \times k} \text{ s.t. } \det A \neq 0 \}$$

$$= \{ (t, A) \in \mathbb{R}^{1 + k \times k} \text{ s.t. } t \det A - 1 = 0. \}$$

$$\subset \mathbb{R}^{1+k^2}.$$

If

$$\pi_1 M = \langle a_i \mid r_j \rangle$$

generators relations

$i = 1, \cdots, N$
$j = 1, \cdots, M$

then what is a representation?

It's a point

$(\phi_1(a_1), \cdots, \phi_N(a_N)) \in GL_k(\mathbb{R})^N$ ← algebraic

satisfying $r_j = I$. ← i.e., satisfying some polynomial condition for each j.

So the space of representations is again a space cut out of some $\mathbb{R}^{\text{huge}}$ by polynomials.

Moreover, conjugation

$$A \longmapsto gAg^{-1}$$

is polynomial in the coordinates (note $\frac{1}{\det A}$ is the quantity t)

so we get an <u>algebraic</u> (i.e. given by polynomials) action on this algebraic space.

The result is what you would say is an algebraic stack.

<u>Rmk</u> As you may know, most algebraic geometry works best assuming you're in an algebraically closed setting — ie, one in which all polynomials have roots. So most people will work w/ $\mathbb{C}$ vec bundles and reps into $GL_k(\mathbb{C})$.

So the lefthand side

$$\{\pi_1 M \to GL_k(\mathbb{R})\}$$

is an algebraic stack. If "stack" is scary, say algebraic "space" w/ singularities

On the other hand, the righthand side

$$\{\text{Flat } (E\to M, \nabla)\}\Big/_{\text{isomorph.}}$$

is harder to think about.

(1) Before modding out by isomorphisms, we see that finding ∇ amounts to solving some differential equation. (We'll see which eq'n, soon.) The space of all ∇ is some giant ∞-dimensional space, and you're cutting out the set of flat connections by some equation, $D\circ\nabla = 0$. A priori we have no idea what this set looks like.

(2) The space of all bundle isomorphisms is huge, but we see that the condition of $\overrightarrow{\text{preserving}}$ flat connections must put some constraints.

(3) Regardless, we might feel that we have little sense of what this righthand space looks like without doing some serious analysis.

So we see a big philosophical pay-off: based on the <u>algebraic</u> nature of lefthand side, we should expect whatever space is on the right to be finite-dimensional. This a priori doesn't seem obvious from the purely "calculus" description.

<u>Warning</u> The bijection alone doesn't tell you how the sides are related as <u>spaces</u>. So this comment