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A general philosophy of math is that functions from X to Y should inherit properties of Y.

<u>ex</u>: $\{f: X \to Y\}$ is a group/ring if Y is.

$\Gamma(T^*M \otimes E)$ is a vec space and a $C^\infty(M)$ module.

has 0
addition
scaling

has $C^\infty(M)$ action.

You can check $\{\text{connections } \nabla\}$ inherits no algebraic structure of $\Gamma(T^*M \otimes E)$. You should interpret this as a reflection of how curved, or non-linear, the Leibniz rule is.

<u>Ex</u>: If ∇_1, ∇_2 are cnxns, $(\nabla_1 + \nabla_2)(fs) = 2df \otimes s + f(\nabla_1 + \nabla_2)s$.

So $\nabla_1 + \nabla_2$ isn't a cnxn.

Likewise for $t\nabla_1$, $t \in \mathbb{R}$, $f\nabla$, $f \in C^\infty(M)$

However:

<u>Prop'n</u>

① $\forall$ $E \to M$ smth vec bundles, connections exist.

② $\forall$ cnxns ∇_0, ∇_1, $(1-t)\nabla_0 + t\nabla_1$ is a connection.

③ $\forall$ cnxns ∇_1, ∇_2, $\nabla_1 - \nabla_2$ is $C^\infty(M)$-linear.

How to prove Propn ①?

We know connections exist locally, so we should try to "patch together" local connections.

This brings us to (a review of) the idea of partitions of unity.

Recall that given a function

$$f: M \to \mathbb{R},$$

the <u>support</u> of f is

$$\mathrm{supp}(f) := \text{closure}\left(\{p \in M \mid f(p) \neq 0\}\right).$$

<u>Ex</u> $\mathrm{supp}(f: \mathbb{R} \xrightarrow{x^2} \mathbb{R}) = \mathbb{R}$.

<u>Defn</u> A <u>partition of unity</u> on a space M is a collection of functions

$$f_\alpha: M \to \mathbb{R}$$

such that

(1) $\{\mathrm{supp}(f_\alpha)\}_\alpha$ is a locally finite collection, and

$\forall p \in M$, $\exists\ U \subset M$ open w/ $p \in U$ s.t. only finitely many $\mathrm{supp}(f_\alpha)$ intersect U.

(2) $\forall p \in M$,

$$\sum_\alpha f_{(\alpha)}(p) = 1.$$

finite sum by (1).

with $f_\alpha(p) \in [0,1]$.

Defn Let $\{U_\beta\}$ be an open cover of M. We say a partition of unity is <u>subordinate</u> to $\{U_\beta\}$ if $\forall \alpha$, $\exists \beta$ s.t.

$$\operatorname{supp}(f_\alpha) \subset U_\beta.$$

Thm Let M be a paracompact topological space. Then $\forall$ open covers $\{U_\alpha\}$, $\exists$ a partition of unity subordinate to $\{U_\alpha\}$. If M is a smth manifold, the partition of unity can chosen to be smooth.

With this in hand, we can prove the existence of connections.

Pf (That connections exist). Choose a trivializing cover $\{U_\beta\}$ for E, and $\{f_\alpha\}$ a partition of unity subordinate to $\{U_\beta\}$. Also choose a connection ∇_β on each:

$$\nabla_\beta : \Gamma(E|_{U_\beta}) \to \Gamma(T^*U_\beta \otimes E|_{U_\beta}).$$

Define

$$\nabla(s) := \sum_\alpha f_\alpha \nabla_{\beta(\alpha)}(s|_{\beta(\alpha)}).$$

$\forall \alpha$, just arbitrarily choosing $\beta(\alpha)$ s.t. $\operatorname{supp} f_\alpha \subset U_{\beta(\alpha)}$.

i.e.,

$$p \longmapsto \underbrace{\sum_{\alpha}}_{\substack{\text{finite sum since } f_\beta \not\equiv 0 \\ \text{for finitely many } \beta.}} f_\alpha \, \nabla_\beta \big(s(p)\big).$$

Why is this a connection?

$$\begin{aligned}
\nabla(fs) &= \sum_\alpha f_\alpha \nabla_\beta \big(fs|_{U_\beta}\big) \\
&= \sum_\alpha f_\alpha \big(df \otimes s + f \, \nabla_\beta s|_{U_\beta}\big) \\
&= \underbrace{\sum_\alpha f_\alpha \cdot df \otimes s}_{\substack{= 1 \cdot df \otimes s \\ \text{since } \{f_\alpha\} \\ \text{is a partition of unity.}}} + f \cdot \underbrace{\sum_\alpha f_\alpha \nabla_\beta \big(s|_{U_\beta}\big)}_{\text{defn of } \nabla(s).} \\
&= df \otimes s + f \, \nabla(s).
\end{aligned}$$

Pf of ②

$$\big((1-t)\nabla_0 + t\nabla_1\big)(fs)$$

$$= (1-t)\nabla_0(fs) + t\nabla_1(fs)$$

$$= (1-t)\, df\otimes s + t\, df\otimes s$$

$$+ f(1-t)\nabla_0(s) + ft\nabla_1(s)$$

$$= df\otimes s + f\big((1-t)\nabla_0 + t\nabla_1\big)(s). \quad //$$

Pf of ③

$$\nabla_1(fs) - \nabla_2(fs)$$

$$= df\otimes s + f\nabla_1 s$$

$$- df\otimes s - f\nabla_2 s$$

$$= f(\nabla_1 - \nabla_2)s. \quad //$$

Curvature for connections

Defn Fix a vector bundle E over M. We let

$$\Omega^k(M;E) := \Gamma(\Lambda^k T^*M \otimes E).$$

An element is called a differential form w/ values in E.

Rmk How should one think of this? As you know, a k-form is a way to integrate over a k-manifold. More geometrically, a k-form $\alpha \in \Omega^k_{deR}(M)$ can be thought of as a way to take flux. Given any configuration of k vectors in T_pM, α_p assigns a # to these vectors. While we're often told to think of this # as a "signed volume" of some parallelepiped spanned by $v_1, \ldots, v_k$, you may also benefit from thinking of $\alpha(v_1 \wedge \cdots \wedge v_k)$ as some generalized Faraday's law, which measures flux through the plane of $v_1, \ldots, v_k$ in a way sensitive to the orientation + magnitudes of the v_i.

Similarly, you might think of $\alpha \in \Omega^k(M;E)$ as a flux w/ values in some physical field. Given $v_1, \ldots, v_k$, $\alpha_p(v_1 \wedge \cdots \wedge v_k)$ outputs not a #, but some element of E_p — If E_p is the vector space of possible electrical currents at p, or of some other state (i.e., some vector in state space) we measure how the state is fluxed through the infinitesimal entity spanned by the v_i.

Now, given a connection

$$\nabla : \Gamma(E) \longrightarrow \Omega^1(M;E)$$

we can inductively define an operation

$$D : \Omega^k(M;E) \to \Omega^{k+1}(M;E)$$

by demanding the Leibniz rule again:

$$D(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \nabla s.$$

What does this mean? Locally, we have a finite sum $\sigma = \sum \beta_i \otimes t_i$ for $\beta_i \in \Omega^k(M)$ and $t_i \in \Gamma(E)$. And $\alpha \wedge \sigma := \sum (\alpha \wedge \beta_i) \otimes t_i$.

Here, we have $\alpha \in \Omega^k(M)$
$s \in \Gamma(E)$.

Ex Fix $M = \mathbb{R}^n$ and $E = \underline{\mathbb{R}}$, and let ∇ be the usual connection

$$C^\infty(M) = \Gamma(\underline{\mathbb{R}}) \longrightarrow \Omega^1_{deR}(M)$$
$$f \longmapsto df.$$

Then
$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge ds$$

for $\alpha \in \Omega^1_{deR}(M)$, $s \in \Gamma(\underline{\mathbb{R}}) = C^\infty(M)$ means that

$$\nabla\left(\sum_{i=1}^n (\alpha_i dx_i) \otimes s\right) = \sum_{i,j} \left(\frac{\partial \alpha_i}{\partial x_j} dx_j \wedge dx_i\right) s + (-1)\left(\sum \alpha_i dx_i\right) \wedge \left(\sum \frac{\partial s}{\partial x_j} dx_j\right).$$

$$= \sum_{i,j} \left(\frac{\partial \alpha_i}{\partial x_j} s + \alpha_i \frac{\partial s}{\partial x_j}\right) dx_j \wedge dx_i$$

$$= \sum_{i,j} \frac{\partial}{\partial x_j}(s\alpha_i) dx_j \wedge dx_i$$

$$= d_{deR}(s\alpha).$$ So we recover the usual deRham differential for Ω^1!

Exer Show that $\forall$
$\alpha \in \Omega^k_{deR}(M)$,
$\beta \in \Omega^l(M;E)$,

$$D(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge D\beta.$$

Pf Locally, write $\beta = \sum_{\text{FINITE}} \beta_i \otimes s_i$ for $\beta_i \in \Omega^l(M;E)$ and $s_i \in \Gamma(E)$.

Then

$$D\left(\sum (\alpha \wedge \beta_i) \otimes s_i\right)$$
$$= \sum d(\alpha \wedge \beta_i) \otimes s_i + (-1)^{k+l} \sum (\alpha \wedge \beta_i) \wedge \nabla s_i$$
$$= d\alpha \wedge \left(\sum \beta_i \otimes s_i\right) + (-1)^{|\alpha|} \alpha \wedge \left(\sum d\beta_i \otimes s_i\right) + (-1)^{|\alpha|} \alpha \wedge \left((-1)^{|\beta|} \sum \beta_i \wedge \nabla s_i\right)$$
$$= d\alpha \wedge \beta + (-1)^k \alpha \wedge \nabla\beta. \quad //$$

As we know, the deRham differential isn't useful solely for the Leibniz rule. It also satisfies

$$d^2 = 0 \quad (\text{ie, } d^k \circ d^{k-1} = 0 \ \forall\, k).$$

<u>Defn</u> A sequence of abelian groups A^k, $k \in \mathbb{Z}$ together with homomorphisms

$$d^k : A^k \to A^{k+1} \quad k \in \mathbb{Z}$$

such that

$$d^{k+1} \circ d^k = 0 \quad \forall\, k$$

is called a <u>cochain</u> <u>complex</u>.

We'll often write this data as (A, d) or $A^\bullet$, or $(A^\bullet, d)$.

If each A^k is a vec space over $\mathbb{R}$ and if each d^k is an $\mathbb{R}$-linear map, then we call $(A^\bullet, d)$ a <u>cochain complex over $\mathbb{R}$</u>.

<u>Ex</u> $A^k = \Omega^k_{deR}(M)$ w/ $d^k = d_{deR}$ gives deRham cohomology of M.

<u>Defn</u> Given a cochain complex, the group (vector space)

$$H^k(A^\bullet) := \ker d^k / \operatorname{im} d^{k-1}$$

is called the k^{th} cohomology group (vector space) of $A^\bullet$.

An obvious question to ask, then, is when

$$\big(\Omega^k(M;E),\ D\big)$$

$$D^0 := \nabla$$

defines a cochain complex.

It turns out that just verifying

$$D^1 \circ \nabla = 0$$

suffices.

Exer

$$D^2 = 0 \iff D^1 \circ \nabla = 0$$

Pf. Suffices to prove this on elements of the form $\alpha \otimes s \in \Omega^k(M;E)$; a general section is locally a finite linear combination of $\alpha_i \otimes s_i$, and if a section ∇^2 is locally zero, it is zero.

$$
\begin{aligned}
D^2(\alpha\otimes s) &= D\big(d\alpha \otimes s + (-1)^{|\alpha|}\alpha\wedge\nabla s\big)\\
&= \overbrace{d^2}^{=0}\alpha\otimes s + (-1)^{|\alpha|+1} d\alpha\wedge Ds\\
&\quad + (-1)^{|\alpha|} d\alpha \wedge Ds + (-1)^{2|\alpha|}\alpha\wedge D(\nabla s)
\end{aligned}
$$

cancel

$$= \alpha \wedge D\nabla s. \quad //$$

So you see, everything is just a consequence of the Leibniz rule.