Some remarks:

<u>Rmk</u> Let $\pi: E \to M$ be a smth vector bundle, and let s_1, s_2 be sections. Note that we can define a new section

$$s_1 + s_2 : M \longrightarrow E$$
$$p \longmapsto s_1(p) + s_2(p).$$

Addition is defined fiber-wise b/c each E_p has structure of a vec space.

Likewise, given $f \in C^\infty(M)$, we can define a new section

$$f \cdot s : M \longrightarrow E$$
$$p \longmapsto f(p) \cdot s(p)$$

where fiber-wise, scaling of $s(p)$ by a real number $f(p)$ is defined by the vector space structure ordained on E_p. This shows

<u>Prop'n</u> $\forall$ smth vec bundles $E \xrightarrow{\pi} M$, $\Gamma(E)$ is a module over $C^\infty(M)$.

Defn Let $\pi: E \to M$ be a smooth vector bundle. Then a <u>connection</u> on E is a map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that the Leibniz rule holds.

ie, $\forall$ $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$\nabla(fs) = df \otimes s + f \nabla s.$$

Heuristically:

- $(f\nabla s)(p)$ depends only on <u>value</u> $f(p)$ and on <u>rate of change</u> of s.
- $(df \otimes s)(p)$ depends on <u>rate of change</u> of f and <u>value</u> of s at p.

$\nabla(fs)$ depends on both.

Let's parse this notation.

- If $f \in C^\infty(M)$, $s \in \Gamma(E)$, fs is the section of E defined previously (via the $C^\infty(M)$-action). So it makes sense to apply ∇ to fs.
- $df \in \Omega^1(M) = \Gamma(T^*M)$. So $\forall$ $p \in M$, $df(p) \in T_pM^\vee$ and $s(p) \in E_p$.

  $df \otimes s$ is the section of $T^*M \otimes E$ that sends
  $$p \longmapsto df(p) \otimes s(p) \in T_pM^\vee \otimes E_p.$$
- Likewise, $\nabla s \in \Gamma(T^*M \otimes E)$; since it's a section of a bundle, it makes sense to scale it by f. This is the $f\nabla s$ term.

Some obvious but important remarks:

Rmk NOT every section of $T^*M \otimes E$ can be written in the form $\alpha \otimes s$ for some $\alpha \in \Omega^1(M)$ $s \in \Gamma(E)$. This is visible already at the level of fibers: The fiber

$$(T^*M \otimes E)_p = T_p^*M \otimes E_p$$

is isomorphic to the vec. space of

$$\underset{\text{rank}E}{k} \times \underset{\dim M}{m} \text{ matrices},$$

and a matrix of the form $\alpha(p) \otimes s(p)$ can have at most one non-zero entry in an appropriate basis. ie, such matrices have at most rank 1.

Here's a preliminary proposition you've probably intuited already:

Prop'n A rank k bundle $E \xrightarrow{\pi} M$ is a trivial bundle iff $\exists$ k sections

$$s_i : M \longrightarrow E$$

s.t. $\forall p \in M$, $\{s_i(p)\}$ are linearly independent in E_p.

Rmk A section of $\Gamma(T^*M \otimes E)$ really does measure rate of change. Fiberwise: given $v \in T_pM$, and $\sum \alpha_i \otimes s_i \in (T^*M \otimes E)_p$

$$= T_pM^\vee \underset{\mathbb{R}}{\otimes} E_p.$$

we get

$$\begin{array}{ccc} \sum \overbrace{\alpha_i(v)}^{\text{a number}} \otimes s_i & \in & \mathbb{R} \underset{\mathbb{R}}{\otimes} E_p \\ \updownarrow & & \cong \\ \sum \alpha_i(v) s_i & & E_p \end{array}$$

ie, if you move in v direction, you change $\sum \alpha_i(v) s_i$ along E_p.

Pf

If $E \overset{f}{\cong} \underline{\mathbb{R}}^k$, $\exists$ sections

$$s_i : p \longmapsto (0, \cdots, 0, 1, 0, \cdots, 0) \overset{f^{-1}}{\longmapsto} s_i(p)$$

i^{th} spot.

Since f is a linear $\cong$ fiberwise, $\{s_i(p)\}$ are lin. ind. since $\{(0, \cdots, 0, 1, 0, \cdots, 0)\}$ are.

Conversely, given $s_i : M \to E$, define

$$f : E \longrightarrow \underline{\mathbb{R}}^k$$
$$v \longmapsto (\pi(v), a_1(v), \cdots, a_k(v))$$

where a_i are the unique real #'s such that

(using that $s_i(p)$ form a basis for E_p.)

$$v = \sum a_i(v) s_i(p) \in E_p.$$

You can check this is smooth — in local coordinates, you can choose a basis for $U \times \mathbb{R}^k \cong E|_U$, and the a_i are obtained by entries of a matrix whose columns are the (local expression of) s_i. Since s_i are smooth, the entries of the matrix are smooth.

You can also check that the inverse map $\underline{\mathbb{R}}^k \xrightarrow{f^{-1}} E$ is smooth (since the map

$$GL_k \longrightarrow GL_k \text{ is smooth}). \text{ So } f \text{ is}$$
$$A \longmapsto A^{-1}$$

indeed a diffeomorphism. //

What do connections on a trivial bundle look like? (This also illustrates what connections look like in local coordinates.)

Prop'n Let $E \xrightarrow{\pi} M$ be a trivial bundle of rank k. Fix linearly independent sections $s_1, \ldots, s_k$. Then any assignment

$$s_i \longmapsto \sigma_i \in \Gamma(T^*M \otimes E)$$

defines a unique connection on E.

Pf Any section $s \in \Gamma(E)$ can be written

$$s = \sum_{i=1}^{k} f_i s_i, \quad f_i \in C^\infty(M).$$

$$p \longmapsto \sum f_i(p) s_i(p).$$

We define

$$\nabla(s) := \sum_{i=1}^{k} df_i \otimes s_i + \sum_{i=1}^{k} \sigma_i. \quad //$$

Rmk Note s_j cannot be written as a $C^\infty(M)$-linear combination of the s_i for $i \neq j$ by linear independence. So there is no "check" of the Leibniz rule for the assignment $s_i \mapsto \sigma_i$. This is a common trick in algebra — you can freely define operations on whole gadgets by creating an arbitrary one on a convenient subgadget.

Rmk The "stupid" assignment

$$\Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$
$$s \longmapsto 0$$

is NOT a connection. For then

$$\nabla(fs) = df \otimes s + f\nabla s = df \otimes s$$

and generally, $df \otimes s \neq 0$.

So the set of cnxns won't form a vec space, but it'll be next best thing: Affine.

Ex Let $E = M\times\mathbb{R}$.

Choose the section

$$s_1 : p \longmapsto (p,1) \in M\times\mathbb{R}.$$

Define σ_1 to be the zero section

$$p \longmapsto 0 \in T_pM^\vee \otimes \mathbb{R}.$$

$$p \longmapsto (0,0) \in T_pM^\vee \times \mathbb{R}.$$

What connection does this define?

$$\nabla(f) = \nabla(f\cdot s_1) := df\otimes s_1 + f\otimes\sigma_1$$

$$= df\otimes s_1 \qquad p\longmapsto df(p)\otimes 1 \in T_p^*M\otimes\mathbb{R} \cong T_p^*M.$$

$$= df$$

But you well could have chosen an arbitrary σ_1, and gotten another connection.

Propn ① Given any vector bundle E, a connection exists.

Propn ② Moreover, if ∇_0 and ∇_1 are connections,

$$t\nabla_0 + (1-t)\nabla_1$$

is also a connection, $\forall\, t\in\mathbb{R}$. ← the space of connections is convex.