Sept 26, 2014

Where are we going?

Goal Construct invariants of vector bundles

$$E \rightsquigarrow p_i(E)$$

Application

$\forall\ E \xrightarrow{\pi} M$, we'll associate elements

$$p_i(E) \in H^{4i}_{deR}(M)$$

called Pontrjagin classes of E.

So these invariants aren't numbers, they're elements of cohomology.

eg. if $p_i(E) \neq p_i(F) \in H^{4i}_{deR}$, we know $E \ncong F$.

In particular, $\forall$ M, we can look at $p_i(TM)$.

If $\dim M = 4n$, we can even integrate products of these $p_i(TM)$ over M.

$$\int_M \prod (p_i)^{d_i} \quad , \quad \sum 4id_i = 4n.$$

These #'s are called the Pontrjagin numbers of M. $\rightsquigarrow$ Invariants of 4n-dimensional manifolds!

How sensitive are they?

Thm If $\exists$ a cobordism from M to M', their Pontrjagin numbers are the same.

ie, $\exists$ W st $\partial W = M \sqcup M'$!

in fact, Pontrjagin and "Stiefel-Whitney" numbers completely determine oriented cobordism class!

Outline of construction of characteristic classes

(1): We'll define a notion of a connection on a vector bundle $E \xrightarrow{\pi} M$.

This is a way (ie, an arbitrary choice) to take derivatives of sections

$$s: E \to M.$$

We'll see that this amounts to an assignment

the connection

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

which way are you moving?

give me a section

here's how your section is changing in the direction of your movement.

An example: If $E = \underline{\mathbb{R}}$, the trivial bundle, then a map

$$\Gamma(TM) \times C^\infty(M) \longrightarrow C^\infty(M)$$

is given by

$$(X, f) \longmapsto X(f)$$

@ p. compute directional derivative of f in direction of X_p.

(2) Given a connection ∇, we can define the notion of <u>curvature</u>.

It is a function

$$R: \Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$
$$(X, \quad Y, \quad s) \mapsto \frac{1}{2}\left\{ \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \right\}.$$

Much of Riemannian geometry is spent trying to understand this quantity. We'll spend some time on it, too. For now, you can see that this quantity detects:

is the non-commutativity of ∇. due only to the non-commutativity of vector fields?

- A small miracle will happen, which is that this assignment is actually a 2-form:

$\Gamma(T^*M \otimes End(E))$

$$R \in \Omega^2(M; End(E))$$

with values in $End(E)$.

the vector bundle given by the functor

$$(Vect^{fd}_{\mathbb{R}})^{isom} \to Vect^{fd}_{\mathbb{R}}$$
$$E \longmapsto End(E)$$
$$\left(E \xrightarrow[isom]{h} E'\right) \longmapsto End(E) \longrightarrow End(E')$$

$$\left(E \xrightarrow{f} E\right) \longmapsto \begin{array}{ccc} E & \xrightarrow{f} & E \\ \uparrow h^{-1} & & \downarrow h \\ E' & \longrightarrow & E' \end{array}$$

- But once we have such a thing, we're in great shape. Why?

If the transition functions for E are given by $g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}^n)$ the transition functions for $\mathrm{End}(E)$ are given by $g_{\alpha\beta} - g_{\alpha\beta}^{-1}: U_\alpha \cap U_\beta \to GL(\mathrm{End}(\mathbb{R}^n))$

$$A \mapsto g_{\alpha\beta} A g_{\alpha\beta}^{-1}.$$

(3)

In other words, if you can associate a real number to each matrix which is invariant under conjugation, then from R, we can define a 2-form on M (w/ $\mathbb{R}$-values).

Ex $\mathrm{tr}(R) \in \Omega^2(M;\mathbb{R})$.

We'll investigate other ways to get numbers like trace out of a matrix.

Here's a sample theorem:

Thm · $d(\mathrm{tr} R) = 0$. (So $\mathrm{tr} R$ defines an element of $H^2_{deR}(M)$.)

· $[\mathrm{tr} R_\nabla] = [\mathrm{tr} R_{\nabla'}]$ in $H^2_{deR}(M)$.

↑ some other connection

(so $[\mathrm{tr} R]$, as an element of $H^2_{deR}(M)$, is an invariant of $E \xrightarrow{\pi} M$ itself!)

As it turns out,
tr never gives an interesting invariant. But other functions (like det) do give interesting invariants.

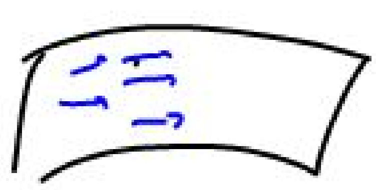
An example of ∇:

How would you try to <u>define</u> derivatives of a vector bundle?

Basic example would be a surface in $\mathbb{R}^3$:

$$\Sigma \subset \mathbb{R}^3.$$

Suppose you're given some C^∞ vector field X on Σ.



and a vector $v \in T_p\Sigma$.

How to measure "rate of change of X" in direction of $\vec{v}$?

Naive attempt: X is a C^∞ fxn

$$\Sigma \longrightarrow T\mathbb{R}^3$$

(since every tangent vector to Σ is a tangent vector to $\mathbb{R}^3$)

so extend to

$$\underline{X} : \underbrace{U}_{\text{some open set about } \Sigma.} \longrightarrow T\mathbb{R}^3$$

Then

$$\underline{X} : \mathcal{U} \longrightarrow T\mathcal{U} \subset T\mathbb{R}^3$$

can be written

$$\underline{X} = \sum f_i \frac{\partial}{\partial x_i}$$

where

$$f_i : \underline{\mathcal{U}} \to \mathbb{R}$$

are smooth. So just define

$$\overline{\nabla}_v (X) := \sum \underbrace{D_v f_i}_{\substack{\text{usual directional derivative} \\ \text{in } v \text{ direction (in } \mathbb{R}^3)}} \frac{\partial}{\partial x_i}\Big|_{\Sigma}$$

In general,

$$\overline{\nabla}_v (X)$$

need not be always tangent to Σ. But we can always project, and take
(See example.)

$$\nabla_v (X) := \Pi_{T\Sigma} \overline{\nabla}_v (X).$$

$$\Pi_{T\Sigma} : T\mathbb{R}^3\big|_{\Sigma} \longrightarrow T\Sigma$$

project vector to components parallel to Σ.
(Using metric everywhere)

This is an example of a connection.

Ex Why isn't $\overline{\nabla}(\underline{X})$ tangent to Σ?

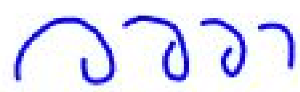
Let

$$\mathbb{R}\times\mathbb{R} \xhookrightarrow{\overline{\underline{X}}} \mathbb{R}^3$$

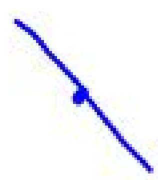
$$(s,\ t) \longmapsto (s\cos t,\ s\sin t,\ t),$$

ex,

s fixed: spiral

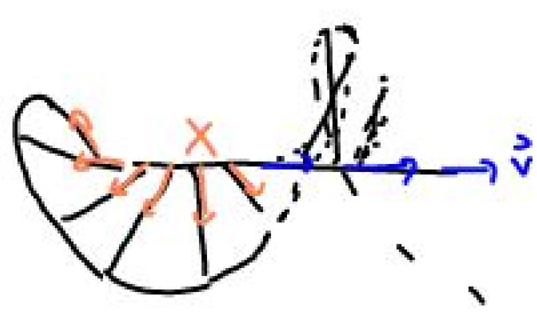


t fixed: line in plane $z = t$ slope in direction of $\sin t, \cos t$.



Consider vector field

$$X : (s,t) = (\cos t, \sin t, 0) + \overline{\underline{X}}(s,t).$$



and let $\vec{v}$ be $(0,0,1)$ at some point w/ $s=0$. Then

$$\overline{\nabla}_{\vec{v}}(X) = \frac{\partial}{\partial t}(\cos t, \sin t, 0)$$

$$= (-\sin t, \cos t, 0)$$

which is $\perp$ to $T_p\Sigma$.

$$T_p\Sigma = \text{image}(\overline{\underline{X}}).$$