Wed, Sept 24, 2014

Question before class:

Let's say you choose a sub-vector space

$$F_p \subset E_p \quad \forall p \in M$$

(Here, $E \xrightarrow{\pi} M$ is a vec. bundle and

$$E_p := \pi^{-1}(p).)$$

How do you know when

$$F := \bigcup_{p \in M} \{p\} \times F_p$$

is a vector bundle over M?

Two ways (at least):

(1) In a trivializing n-hood $U \subset M$, can you choose a <u>smooth</u> basis of sections

$$s_1, \dots, s_k : M \longrightarrow E$$

s.t. $s_i(p) \in F_p \quad \forall p \in M$,

$$k = \dim F_p ?$$

(2) Can you exhibit F as image/kernel of a smooth bundle map w/ constant rank?

Review of last time:

Functors

How to construct vect bundles out of functors.

Let $\mathrm{Vect}_{\mathbb{R}}^{f.d}$ be category of fin-dim vector spaces $/\mathbb{R}$.

$$\hom(X,Y) = \{ \text{linear } f: X \to Y \}.$$

We don't topologize X, Y, but we can choose a linear isom.

$$\hom(X,Y) \cong \mathbb{R}^{(\dim X)(\dim Y)}$$

to give $\hom(X,Y)$ structure of a smth mfld.

Likewise, if

$$\mathcal{C} = \mathrm{Vect}_{\mathbb{R}}^{f.d.} \times \mathrm{Vect}_{\mathbb{R}}^{f.d.}$$

where
- $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\mathrm{Vect}_{\mathbb{R}}^{f.d} \times \mathrm{ob}\mathrm{Vect}_{\mathbb{R}}^{f.d.}$

$= \{ (X_1, X_2) \mid X_i$ are finite-dim vect spaces $/\mathbb{R} \}$.

- $\hom((X_1,X_2),(Y_1,Y_2)) = \hom(X_1,Y_1) \times \hom(X_2,Y_2)$

$= \{ (f_1, f_2)$ s.t. $f_i : X_i \to Y_i$ is linear $\}$

we can endow $\hom((X_1,X_2),(Y_1,Y_2))$ w/ smth mfld structure $\forall (X_1,X_2),(Y_1,Y_2)$.

Rmk If $F: \mathcal{C} \to \mathcal{D}$ is a functor, and $\mathcal{C}, \mathcal{D}$ are $\mathrm{Vect}^{fd}_{\mathbb{R}}$, or $\mathrm{Vect}^{fd}_{\mathbb{R}} \times \mathrm{Vect}^{fd}_{\mathbb{R}}$, or $(\mathrm{Vect}^{fd}_{\mathbb{R}})^{op}$, the function

$$\hom_{\mathcal{C}}(X,Y) \xrightarrow{F} \hom_{\mathcal{D}}(F(X), F(Y))$$

is often a <u>linear</u> function. (These hom-sets are also vector spaces for the above choices of $\mathcal{C}, \mathcal{D}$.) Linear functions are smooth, so such "linear" F are always Mfld-enriched functors.

Ex of F:

$\mathrm{Vect}^{fd}_{\mathbb{R}} \to \mathrm{Vect}^{fd}_{\mathbb{R}}$

$E \mapsto E^{\otimes k}$

or

$E \mapsto \Lambda^k E$

or

$E \mapsto \hom(E, \mathbb{R})$

or

$(E_1, E_2) \mapsto E_1 \oplus E_2$

or

$(E_1, E_2) \mapsto E_1 \otimes E_2$.

Construction Fix $E \xrightarrow{\pi} M$ a C^∞ vec bundle.

Fix a functor $F: \mathrm{Vect}^{fd}_{\mathbb{R}} \to \mathrm{Vect}^{fd}_{\mathbb{R}}$ which is Mfld-enriched.

We construct a new vector bundle

$$F(E)$$

as follows:

As a set, $F(E) = \bigcup_{p \in M} \{p\} \times F(E_p)$

since F is a functor, $\forall$ vec spaces E_p, it assigns a vec space $F(E_p)$.

Topologize so that $\forall$ trivializing n-hood $U \subset M$, the map

$$F(E)|_U = \bigcup_{p \in U} \{p\} \times F(E_p) \xrightarrow{id_U \times F(\phi)} U \times F(\mathbb{R}^n)$$

is a homeomorphism.

U trivializing $\iff$ $\exists$ bundle isom $E|_U \xrightarrow[\phi]{\cong} U \times \mathbb{R}^n$.

$\forall p \in U$, $\phi(p,-)$ is a linear isom $\phi_p : E_p \xrightarrow{\cong} \mathbb{R}^n$.

$F(\phi)$ is the map $(p, \vec{v}) \mapsto (p, \underbrace{F(\phi_p)}(\vec{v}))$.

F is a functor, so assigns linear $\cong$ to ϕ_p.

To give $F(E)$ an atlas, we can assume (or only consider) those trivializing neighborhoods $U_\alpha \subset M$ that are diffeomorphic to an open subset of $\mathbb{R}^{\dim M} = \mathbb{R}^m$.

The transition functions for $F(E)$ are then given by

$$U_\alpha \cap U_\beta \times F(\mathbb{R}^n) \xrightarrow{F(\phi_\alpha)^{-1}} F(E)\big|_{U_\alpha \cap U_\beta} \xrightarrow{F(\phi_\beta)} U_\alpha \cap U_\beta \times F(\mathbb{R}^n)$$

Since each map respects the $U_\alpha \cap U_\beta$ coordinate, this is the same data as a map

$$U_\alpha \cap U_\beta \xrightarrow{F(g_{\alpha\beta})} GL(F(\mathbb{R}^n)) = \text{invertible linear maps } F(\mathbb{R}^n) \to F(\mathbb{R}^n)$$

has a smth structure since it's an open subset of $\hom(F(\mathbb{R}^n), F(\mathbb{R}^n))$.

why is this smooth? By F being Mfld-enriched.

We have a factoring

$$\begin{tikzcd} U_\alpha \cap U_\beta \arrow[rr, "F(g_{\alpha\beta})"] \arrow[dr, "g_{\alpha\beta}"'] & & GL(F(\mathbb{R}^n)) \\ & GL(\mathbb{R}^n) \arrow[ur, "F"'] & \end{tikzcd}$$

where $g_{\alpha\beta}$ is smooth since E is a smth vec bundle, and F is smth since it's induced by the map on hom-sets of a Mfld-enriched functor.

<u>Rmk</u> We're using that F respects composition to relate $F(\phi_\beta \circ \phi_\alpha^{-1})$ to $F(g_{\alpha\beta})$.

Rmk If F is contravariant,

so

$$\left(\mathrm{Vect}_{\mathbb{R}}^{fd}\right)^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}^{fd}$$

the maps $F(\phi)$ go the wrong way:

$$E|_U \xrightarrow{\phi} U \times \mathbb{R}^n \quad \overset{F}{\rightsquigarrow} \quad F(E)|_U \xleftarrow{F(\phi)} U \times F(\mathbb{R}^n).$$

wrong way!

So we define trivializing maps to be the inverse of $F(\phi)$:

$$F(E)|_U \xrightarrow{F(\phi)^{-1} = F(\phi^{-1})} U \times F(\mathbb{R}^n).$$

So for example, if you work out the transition maps for T^*M, you'll get the inverse matrices of the transition maps for TM.

Defn The cotangent bundle of M is

$$T^*M := \hom(TM, \mathbb{R}).$$

If $F : \mathrm{Vect}^{op} \longrightarrow \mathrm{Vect}$

$E \longmapsto \hom(E, \mathbb{R})$,

then $T^*M = F(TM)$.

A differential k-form, or smooth k-form, is a section of

$$\Lambda^k T^*M.$$

We let

$$\Omega^k := \Omega^k(M) := \Omega^k(M; \mathbb{R})$$

$$:= \Gamma(\Lambda^k T^*M)$$

denote the set of all smooth k-forms.

Ex A 1-form is a choice of $\alpha(p) \in T_p^* M$ $\forall p \in M$.

So it takes a smth vec field and spits out a fnction.

Defn $\forall p \in \mathbb{R}^n$, let

$$dx_i|_p \in (T_p\mathbb{R}^n)^\vee = \hom(T_p\mathbb{R}^n, \mathbb{R}) \quad , \quad i=1,\dots,n,$$

denote the dual vector to $\frac{\partial}{\partial x_i}\Big|_p \in T_p\mathbb{R}^n$.

This means

$$dx_i|_p \left(\frac{\partial}{\partial x_j}\Big|_p\right) = \begin{cases} 1 & i=j \\ 0 & \text{otherwise.} \end{cases}$$

Defn Let

$$dx_i \in \Gamma(T^*\mathbb{R}^n)$$

denote the section

$$p \longmapsto dx_i|_p.$$

Likewise,

$$\frac{\partial}{\partial x_i} \in \Gamma(T\mathbb{R}^n)$$

denotes the section

$$p \longmapsto \frac{\partial}{\partial x_i}\Big|_p.$$

Ex A 2-form α on $\mathbb{R}^n$ is a choice of C^∞ fxns

$$f_{i,j} \qquad \begin{array}{l} i<j, \\ i,j \in \{1,\cdots,n\} \end{array}$$

where

$$\alpha = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j.$$

ie,

$$\alpha(p) = \sum_{i<j} \underbrace{f_{i,j}(p)}_{\substack{\text{a} \\ \text{real \#}}} \underbrace{dx_i|_p \wedge dx_j|_p}_{\substack{\text{an element} \\ \text{of } \Lambda^2 T^*_p \mathbb{R}^n}}$$

Recollections on H^*_{deR}:

$\exists!$ collection of $\mathbb{R}$-linear maps

$$d^k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \qquad k \geq 0$$

satisfying

(1) $d^2 = 0$ (i.e., $d^{k+1} \circ d^k = 0 \ \forall k \geq 0$)

This means $(\Omega^\bullet(M), d)$ is a cochain complex.

(2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in \Omega^k$, $\beta \in \Omega^l$.

So d is a graded derivation for the multiplication on $\Omega^\bullet(M)$.

(3) $\forall f \in \Omega^0(M) = C^\infty(M)$, df is the usual differential of f.

ie, $df(X) = X(f) \quad \forall X \in \Gamma(TM)$.

This unique d is the <u>deRham</u> <u>differential</u>,

and is sometimes written

$$d_{deR}.$$

On $\mathbb{R}^n$, for instance,

$$d_{deR}\left(\sum f_{i,j}\, dx_i \wedge dx_j\right)$$

$$= \sum \frac{\partial f_{i,j}}{\partial x_k}\, dx_k \wedge dx_i \wedge dx_j,$$

on 2-forms.

(I'm going quickly because I've been told you've seen this before.)

<u>Defn</u>

$$H^k_{deR}(M) := \mathrm{Kernel}(d^{k+1}) \big/ \mathrm{Image}(d^k).$$

$H^\bullet_{deR}(M)$ is the algebra w/ multiplication

$$H^k_{deR}(M) \otimes H^\ell_{deR}(M) \longrightarrow H^{k+\ell}_{deR}(M)$$

induced by $\wedge$ of forms:

$$[\alpha] \otimes [\beta] \longmapsto [\alpha \wedge \beta].$$

well-defined since $d(\alpha\wedge\beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$.

This has the property that

$$[\alpha\wedge\beta] = (-1)^{k\ell}[\beta\wedge\alpha] \quad \text{for } \alpha \in \Omega^k, \ \beta \in \Omega^\ell.$$

Q: Why

$\Lambda^{\bullet}(TM)$? ← free graded commutative alg

Why not

$\bigotimes^{\bullet}(TM)$? ← free associative algebra

A: I don't know. Maybe because graded commutative things are just easier to study. I don't know what interest

$$H^{\bullet}(\otimes^{\bullet}(TM))$$

has.

Next time: What structures of

$$\Lambda^k TM$$

remain in an arbitrary vector bundle E?