Mon, Sept 22, 2014

We'll learn soon about characteristic classes But I want to give a more systematic picture for creating vector bundles.

The general picture one might conjecture is:

Given $E \xrightarrow{\pi} M$ a vector bundle, consider charts

a vector space; $E_p = \pi^{-1}(p)$.

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^n \arrow[dr] & \pi^{-1}(U_\alpha) = \bigcup_{p \in U_\alpha} \{p\} \times E_p \arrow[l, "\phi_\alpha"'] \arrow[d] \\ & U_\alpha \end{tikzcd}$$

(Note $E \cong \coprod_\alpha U_\alpha \times \mathbb{R}^n \Big/ \phi_\beta \circ \phi_\alpha^{-1}$)

Well, what if we have an assignment (aka functor)

- which replaces each vector space E_p with $F(E_p) := F_p$
  $\mathbb{R}^n$ with $F(\mathbb{R}^n)$
- every linear isomorphism $\mathbb{R}^n \xleftarrow{\phi_{\alpha,p}} \{p\} \times E_p$ with $F(\mathbb{R}^n) \xleftarrow{F(\phi_{\alpha,p})} \{p\} \times F_p$

another linear isomorphism

such that

- composition of linear maps is respected, and
- If $U_\alpha \cap U_\beta \xrightarrow{\phi_\beta \circ \phi_\alpha^{-1}} GL(\mathbb{R}^n)$ is C^∞, then $U_\alpha \cap U_\beta \xrightarrow{F(\phi_\beta \circ \phi_\alpha^{-1})} GL(F(\mathbb{R}^n))$ is.

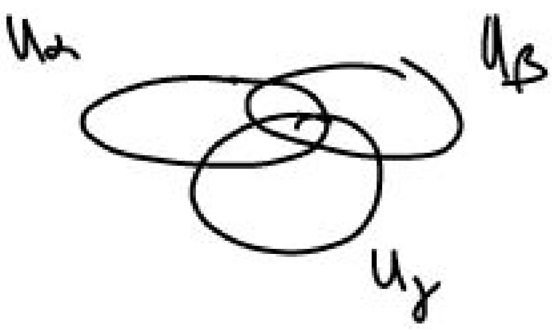
Then $F := \coprod U_\alpha \times F(\mathbb{R}^n) \Big/ F(\phi_\beta \circ \phi_\alpha^{-1})$ defines a new vec bundle over M.

$$U_\alpha \times \mathbb{R}^n \amalg U_\beta \times \mathbb{R}^n \Big/ (x, \vec{v}) \sim (y, \vec{w}) \text{ if}$$

$$y = pr_U \circ \phi_\beta \circ \phi_\alpha^{-1}(x)$$

$$\vec{w} = pr_{\mathbb{R}^n} \circ g_\beta \circ g_\alpha^{-1}(\vec{v})$$

$$= pr_{\mathbb{R}^n} \circ g_{\alpha\beta}(\vec{v})$$



$$\begin{tikzcd}
U_\alpha \times \mathbb{R}^n \arrow[rr, dashed, "g_{\alpha\beta}"] & & U_\beta \times \mathbb{R}^n \arrow[dl, dashed, "g_{\beta\gamma}"] \\
& U_\gamma \times \mathbb{R}^n \arrow[ul, dashed, "g_{\gamma\alpha}"] &
\end{tikzcd}$$

$--\!\!\rightarrow$ means, defined on an open subset.

Note $g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\gamma} = id$. ie,

$\forall$ $p \in U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$g_{\gamma\alpha} \circ g_{\beta\gamma} \circ g_{\alpha\gamma}(p) = id_{\pi^{-1}(p)} : E_p \to E_p.$$

Defn A category $\mathcal{C}$ is the data of:

- A collection $\mathrm{ob}\,\mathcal{C}$ called the objects of $\mathcal{C}$
- $\forall\, X, Y \in \mathrm{ob}\,\mathcal{C}$, a set
$$\hom(X,Y)$$
called the set of morphisms from X to Y,
- $\forall\, X, Y, Z \in \mathrm{ob}\,\mathcal{C}$, a composition map
$$\hom(X,Y)\times\hom(Y,Z) \xrightarrow{\ \circ\ } \hom(X,Z)$$

such that

- $\forall\, X$, $\exists\ id_X \in \hom(X,X)$
s.t. $f \circ id_X = f \ \forall\, f \in \hom(X,Y),\ \forall\, Y$
$id_X \circ f = f \ \forall\, f \in \hom(Y,X)\ \forall\, Y.$

(Note by taking $X=Y$, id_X is unique.)

- Composition is associative. i.e.,

$$\begin{tikzcd}
 & \hom(X,Y)\times\hom(Y,Z)\times\hom(Z,W) \arrow[dl, "id_{\hom(X,Y)}\times\circ"'] \arrow[dr, "\circ\times id_{\hom(Z,W)}"] & \\
\hom(X,Y)\times\hom(Y,W) \arrow[dr, "\circ"'] \arrow[rr, phantom, "\circlearrowleft"] & & \hom(X,Z)\times\hom(Z,W) \arrow[dl, "\circ"] \\
 & \hom(X,W) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (f,g,h) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(f, h\circ g) \arrow[dr, mapsto] & & (g\circ f, h) \arrow[dl, mapsto] \\
 & (h\circ g)\circ f \qquad h\circ(g\circ f) &
\end{tikzcd}$$

i.e.,

$$(h\circ g)\circ f = h\circ(g\circ f).$$

Ex $\forall$ groups/monoids G,

we have a category BG:

- a single object X
- $\hom(X,X) := G$
- Composition is given by the group/monoid multiplication:

$$\begin{array}{ccc} \hom(X,X) \times \hom(X,X) & \xrightarrow{\circ} & \hom(X,X) \\ \| & & \| \\ G \times G & \longrightarrow & G. \end{array}$$

Ex $\mathcal{C} =$ Groups

$\mathrm{ob}\,\mathcal{C} =$ "set" of all groups

$\hom(X,Y) =$ set of all group homomorphisms $f: X \to Y$

Composition, id_X as usual.

Ex $\mathcal{C} = \mathrm{Cob}_n$, category of <u>n-cobordisms</u>.

- $\mathrm{ob}\,\mathcal{C} = \{\text{smth, compact } (n-1)\text{ manifolds}\}/\text{diffeo}$
- $\hom(X,Y) = \Big\{$ smth n-mflds W, possibly non-compact, together w/ a <u>proper</u> map $W \xrightarrow{p} \mathbb{R}$ s.t.
  - $\exists$ diffeos
$$\begin{array}{ccc} p^{-1}(-\infty,0) & \xrightarrow{\cong} & X \times (-\infty,0) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}} & \\ & (-\infty,0) & \end{array}$$
$$\begin{array}{ccc} p^{-1}(1,\infty) & \xrightarrow{\cong} & Y \times (1,\infty) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}} & \\ & (1,\infty) & \end{array}$$
$\Big\} / \mathrm{diff}(W)$ or $\mathrm{diff}(\mathbb{R})$.

Composition

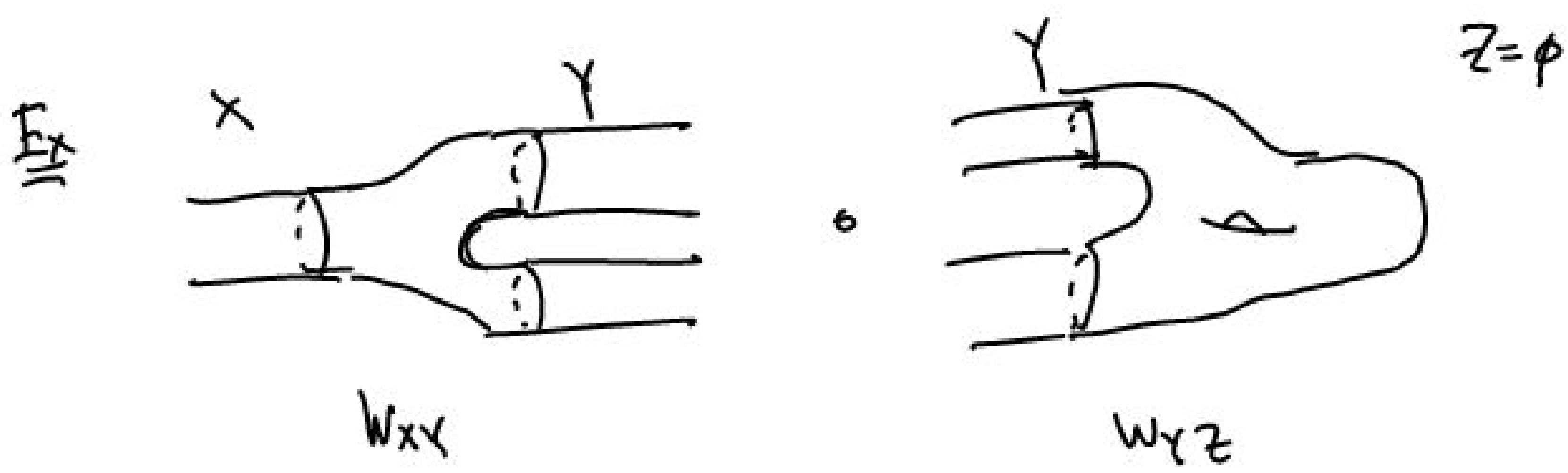
$$\hom(X,Y) \times \hom(Y,Z) \longrightarrow \hom(X,Z)$$
$$(W_{XY}, W_{YZ}) \longmapsto W_{YZ} \circ W_{XY}$$

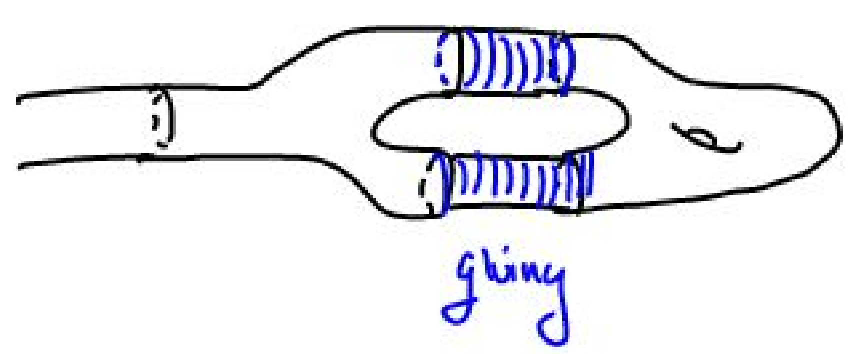
Is given by

$$W_{YZ} \circ W_{XY} := \Big[W_{XY} \bigcup_{\substack{Y \times (1, 1+\varepsilon) \\ \sim Y \times (-\varepsilon, 0)}} W_{YZ} \Big]$$

= diff$/\mathbb{R}$ class of what you obtain by gluing W_{XY} to W_{YZ} along $Y \times$ (open interval).

Ex





Rmk p proper $\Rightarrow$ $p^{-1}([-\varepsilon, 1+\varepsilon])$ is a compact manifold w/ boundary $X \amalg Y$.

Rmk $W_{XY} \bigcup_Y W_{YZ}$ is NOT well-defined as a smooth manifold, while $W_{XY} \bigcup_{Y \times \text{open interval}} W_{YZ}$ is. As an example, every diffeomorphism type of S^7 can be obtained as

$$D^7 \bigcup_{S^6} D^7 .$$

Ex Let $\mathrm{TopVect}_{\mathbb{R}} = \mathcal{C}$

be category where

ob $\mathcal{C}$ = Vector spaces/$\mathbb{R}$
given topology st
scaling + addition
are C^0

hom(X,Y) = C^0 linear maps
$X \to Y$.

Defn A category $\mathcal{C}$ is

- Top-enriched
- or . Mfld-enriched if
  - each hom(X,Y) is given
    the structure of
    a · topological space and
    · manifold
  - $\mathrm{hom}(X,Y) \times \mathrm{hom}(Y,Z) \to \mathrm{hom}(X,Z)$
    is a C^0 map.
    C^∞

In genl, if $\mathcal{D}$ is a "symmetric monoidal" category w/ product $\otimes$, makes sense to speak of $\mathcal{D}$ enriched categories:

- $\forall$ $X,Y \in \mathrm{ob}\,\mathcal{C}$,
  $\mathrm{hom}(X,Y) \in \mathrm{ob}\,\mathcal{D}$
- given maps
  $\mathrm{hom}(X,Y) \otimes \mathrm{hom}(Y,Z) \to \mathrm{hom}(X,Z)$
  in $\mathcal{D}$.

Ex Restrict to finite-dimensional vect spaces w/ Euclidean topology (ie, usual topology) inherited from a choice of linear isomorphism

$$V \cong \mathbb{R}^n.$$

(Topology is indep. of choice of isom.) This category

$$\mathrm{TopVect}_{\mathbb{R}}^{\mathrm{f.d.}} \subset \mathrm{TopVect}_{\mathbb{R}}$$

can be made Mfld-enriched, as

$$\hom(X,Y) \cong \{\dim X \times \dim Y \text{ matrices}\}$$

$$\cong \mathbb{R}^{(\dim X)(\dim Y)}$$

So we endow $\hom(X,Y)$ w/ smooth structure given by a choice of above isomorphisms. (Again, independent of choice.)

Defn A functor from $\mathcal{C}$ to $\mathcal{D}$

is an assignment

- $\forall\ X \in \mathrm{ob}\,\mathcal{C}$,

  $F(X) \in \mathrm{ob}\,\mathcal{D}$

- a function

  $\hom_{\mathcal{C}}(X,Y) \xrightarrow{F} \hom_{\mathcal{D}}(F(X),F(Y))$

  $\forall\ X,Y \in \mathrm{ob}\,\mathcal{C}$.

such that

- $F(id_X) = id_{F(X)}$

- F respects composition:

  $F(g \circ f) = F(g) \circ F(f)$.

If $\mathcal{C}$ and $\mathcal{D}$ are Top- (or Mfld) enriched,

we say F is a Top- (or Mfld-) enriched functor

if $\quad F : \hom(X,Y) \longrightarrow \hom(F(X),F(Y))$

is $\quad C^0 \quad (C^\infty)$.

From hereon, for brevity's sake, let

$$\mathrm{Vect}_{\mathbb{R}}^{f.d.}$$

be category of fin-dim $\mathbb{R}$-vector spaces. We give

$$\hom(X,Y)$$

a smooth stucture via linear isos

$$\hom(X,Y) \cong \hom(\mathbb{R}^{\dim X}, \mathbb{R}^{\dim Y}) \cong \mathbb{R}^{(\dim X)(\dim Y)}.$$

Also, note

$$\mathrm{Vect} \times \mathrm{Vect}$$

objects (X_1, X_2)

$$\hom((X_1,X_2),(Y_1,Y_2)) = \hom(X_1,Y_1) \times \hom(X_2,Y_2)$$

is also Mfld-enriched.

Most functors you can think of are Mfld-enriched!

__Ex__

$$\mathrm{Vect}_{\mathbb{R}}^{fd} \longrightarrow \mathrm{Vect}_{\mathbb{R}}^{fd}$$

$$E \longmapsto E^{\otimes k}$$

$$(E \xrightarrow{f} F) \longmapsto f^{\otimes k}: E^{\otimes k} \to F^{\otimes k}$$

$$v_1 \otimes \cdots \otimes v_k \longmapsto f(v_1) \otimes \cdots \otimes f(v_k).$$

The function

$$\hom(E,F) \longrightarrow \hom(E^{\otimes k}, F^{\otimes k})$$

$$f \longmapsto f^{\otimes k}$$

is C^∞ since its a polynomial function in the matrix entries of f.

· ex: If $E \cong F \cong \mathbb{R}$, $f(v) = av$,

then $f^{\otimes k}(v \otimes \cdots \otimes v) = a^k (v \otimes \cdots \otimes v)$.

__Ex__ Fix A finite-dim vec space.

$$(\mathrm{Vect}_{\mathbb{R}}^{fd})^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}^{fd}$$

$$E \longmapsto \hom(E, A)$$

$$(E \xrightarrow{f} F) \longmapsto \hom(E,A) \xleftarrow{f^*} \hom(F,A)$$

is also Mfld-enriched.

__Ex__ $\mathrm{Vect}_{\mathbb{R}}^{fd} \xrightarrow{\Lambda^k} \mathrm{Vect}_{\mathbb{R}}^{fd}$

$$E \longmapsto \Lambda^k(E)$$

$$E \xrightarrow{f} F \longmapsto \Lambda^k f : v_1 \wedge \cdots \wedge v_k \longmapsto f(v_1) \wedge \cdots \wedge f(v_k).$$