New bundles

Motto: Any functorial way of constructing vector spaces give a way of constructing vector bundles.

Ex

$E, F \rightsquigarrow E \oplus F$ direct sum bundle

$E, F \rightsquigarrow E \otimes F$ tensor product bundle

$E \rightsquigarrow E^\vee$ dual bundle

$E \rightsquigarrow \Lambda^k E$ k^{th} exterior power bundle

What do we need to do to construct a vector bundle?

Make a total space E

a map $E \xrightarrow{\pi} M$

a structure of a $\mathbb{R}$-vector space on each $\pi^{-1}(p)$

allowing for

a cover $\{U_\alpha\}$ of M such that

$\exists$ a bundle isomorphism

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\cong"] \arrow[dr] & U_\alpha \times \mathbb{R}^n \arrow[d] \\
& U_\alpha
\end{tikzcd}$$

Ex Let $E \xrightarrow{\pi_E} M$

$F \xrightarrow{\pi_F} M$

be vector bundles.

Define a set

$$E \oplus F = \bigcup_{p \in M} \{p\} \times \pi_E^{-1}(p) \oplus \pi_F^{-1}(p).$$

has obvious map $\pi_{E\oplus F} : E \oplus F \longrightarrow M$
$(p, \vec{u}, \vec{w}) \longmapsto p$

If U_α a trivializing neighborhood for E,
U_β a " " for F,

with trivialization Φ_α, Φ_β,

$$\begin{tikzcd} \pi_E^{-1}(U_\alpha) \arrow[r, "\cong"] \arrow[dr, "\pi_E"'] & U_\alpha \times \mathbb{R}^n \arrow[d, "pr_1"] \\ & U_\alpha \end{tikzcd} \quad (\Phi_\alpha)$$

$$\begin{tikzcd} \pi_F^{-1}(U_\beta) \arrow[r] \arrow[dr, "\pi_F"'] & U_\beta \times \mathbb{R}^m \arrow[d, "pr_1"] \\ & U_\beta \end{tikzcd} \quad (\Phi_\beta)$$

$$\Phi_\alpha : \bigcup_{p \in U_\alpha} \{p\} \times \pi_E^{-1}(p) \longrightarrow U_\alpha \times \mathbb{R}^n$$
$$(p, \vec{v}) \longmapsto (p, \Phi_{\alpha,p}(\vec{v}))$$

$$\Phi_\beta : \bigcup \{p\} \times \pi_F^{-1}(p) \longrightarrow U_\beta \times \mathbb{R}^n$$
$$(p, \vec{w}) \longmapsto (p, \Phi_{\beta,p}(\vec{w}))$$

we can construct a map

$$\Phi_{\alpha,\beta} : \pi_{E\oplus F}^{-1}(U_\alpha \cap U_\beta) \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n \oplus \mathbb{R}^m$$
$$(p, \vec{v}, \vec{w}) \longmapsto (p, \Phi_{\alpha,p}(\vec{v}), \Phi_{\beta,p}(\vec{w})).$$

Topologize so these are homeomorphisms.

The $\{\Phi_{\alpha,\beta}\}_{\alpha,\beta}$ define an atlas by taking U_α, U_β to be homeomorphic to open subsets of $\mathbb{R}^n$.

What do the transition functions look like? On $U_\alpha \cap U_\beta$ component, they're usual transition maps for M's atlas. On the $\mathbb{R}^n \oplus \mathbb{R}^m$ component, we have linear isomorphisms

$$\{p\} \times \mathbb{R}^n \oplus \mathbb{R}^m \xrightarrow{\phi_{\alpha'\beta'} \circ \phi_{\alpha\beta}^{-1}} \{p\} \times \mathbb{R}^n \oplus \mathbb{R}^m$$

$$(p, \vec{v}, \vec{w}) \longmapsto (p, \phi_{\alpha'} \phi_\alpha^{-1}(\vec{v}), \phi_{\beta'} \phi_\beta^{-1}(\vec{w})).$$

ie, direct sum of transition functions.

$$\begin{pmatrix} \boxed{E} & 0 \\ 0 & \boxed{F} \end{pmatrix}.$$

Likewise for $E \otimes F$ etc.

Defn Let $E \xrightarrow{\pi} M$ be a smth vec bundle, $f : N \to M$ a C^∞ map. Then

$$f^*E := \{(p, \vec{v}) \in N \times E \text{ s.t. } f(p) = \pi(\vec{v})\}.$$

$$\begin{tikzcd}
(p, \vec{v}) \arrow[rr, mapsto] \arrow[ddd, mapsto] & & \vec{v} \\
& f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
& N \arrow[r, "f"] & M \\
p & &
\end{tikzcd}$$

The projection $f^*E \to N$ makes f^*E a vec bundle over N.

Ex $U_\alpha \hookrightarrow M$ trivializing

$$\Rightarrow \begin{tikzcd}
f^{-1}(U_\alpha) \times \mathbb{R}^n \arrow[r] \arrow[d] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
f^{-1}(U_\alpha) \arrow[r, hook] & N \arrow[r, "f"'] & M
\end{tikzcd}$$

so $f^{-1}(U_\alpha)$ trivializing.

Ex Let E_1, E_2 be vector bundles over M. Then $E_1 \times E_2 \to M \times M$ is a vector bundle. If

$$f : M \to M \times M, \quad p \mapsto (p, p),$$

then $f^*(E_1 \times E_2) = E_1 \oplus E_2$.

Ex Let

$$f : S^n \hookrightarrow \mathbb{R}^{n+1}$$

be standard embedding.

Then

$$f^*(T\mathbb{R}^{n+1})$$

is the trivial bundle

$$\underline{\mathbb{R}^{n+1}} := \mathbb{R}^{n+1} \times S^n$$

since f^* pulls back the trivializing n-hood $\mathbb{R}^{n+1} = U$.

Ex $\forall n$,

$$TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}^{n+1}}.$$

trivial bundle

Let $E = \{(x, v)$ s.t. $x \in S^n,$
$v = \lambda x$
for some $\lambda \in \mathbb{R}\}$

$= \{(x, v)$ s.t. $x \in S^n,$
$v \perp T_x S^n\}.$

Then $T_x S^n \oplus T_x E \cong T_x \mathbb{R}^n$

$\forall x$, and $\underline{\mathbb{R}^{n+1}} \cong f^*(T\mathbb{R}^{n+1})$.

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