

Wed, Sept 17, 2014

lets prove

Thm Let M be a smooth manifold. Then

$$T(TM) \cong \text{Der}_{\mathbb{R}}(C^{\infty}(M), C^{\infty}(M))$$

We first prove it for M an open subset of \mathbb{R}^n , where the proof will seem concrete.

How did we define TM ?

Given an atlas, $A_M = \{(U_\alpha, \phi_\alpha)\}_\alpha$

we topologized TM by

gluing together

$$\phi_\alpha(U_\alpha) \times \mathbb{R}^n$$

via transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} \times D(\phi_\beta \circ \phi_\alpha^{-1}).$$

Then

$$\tilde{U}_\alpha = \bigcup_{p \in U_\alpha} \{p\} \times T_p M$$

and
$$\tilde{\phi}_\alpha : (x, v) \mapsto (\phi_\alpha(x), v(x; \circ \phi_\alpha))$$

defined an atlas $\{(\tilde{U}_\alpha, \tilde{\phi}_\alpha)\}_\alpha$

for TM .

In a similar fashion,

let

$$(T_p M)^\vee := \text{hom}_{\mathbb{R}}(T_p M, \mathbb{R})$$

Then one can give

$$T^*M := \bigcup_{p \in M} \{p\} \times T_p M^\vee$$

the structure of a smooth vector bundle. This is the cotangent bundle of M .

Def A section of T^*M is called a smooth 1-form on M .

Rmk \exists map

$$\Gamma(T^*M) \times \Gamma(TM) \rightarrow C^\infty(M).$$

Exer (For home) Prove
smooth structure of TM
is independent of choice
of atlas on M .

We saw TM is a vector
bundle.

Def let $E \rightarrow M$
and $E' \rightarrow M'$

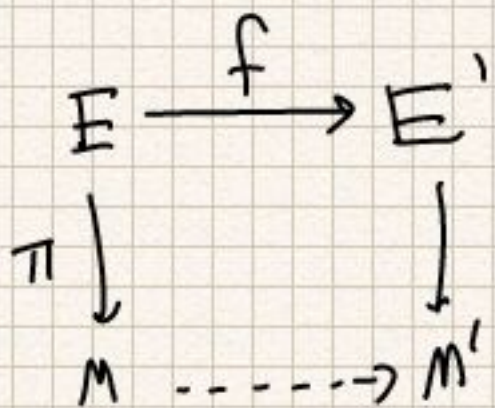
be smooth vector bundles.

A bundle map from E to

E' is a smooth map

$f: E \rightarrow E'$ s.t.

\exists a smooth map



\Leftrightarrow " u, v in
same fiber $\Rightarrow f(u), f(v)$
in same
fiber "

making this diagram commute.

$f|_{\pi^{-1}(p)}$ is a linear map $\forall p \in M$

A (smooth) bundle map f is called a (smooth) bundle isomorphism if f is a diffeomorphism.

— In our class, smoothness is always assumed.

Exer (Home) If f is a diffeomorphism, then f^{-1} is a bundle map.

Prop Let $U \subset \mathbb{R}^n$ be open, and given the standard atlas $\{(U, \phi: U \rightarrow \mathbb{R}^n)\}$

Then \exists a bundle isomorphism

$$TU \cong U \times \mathbb{R}^n$$

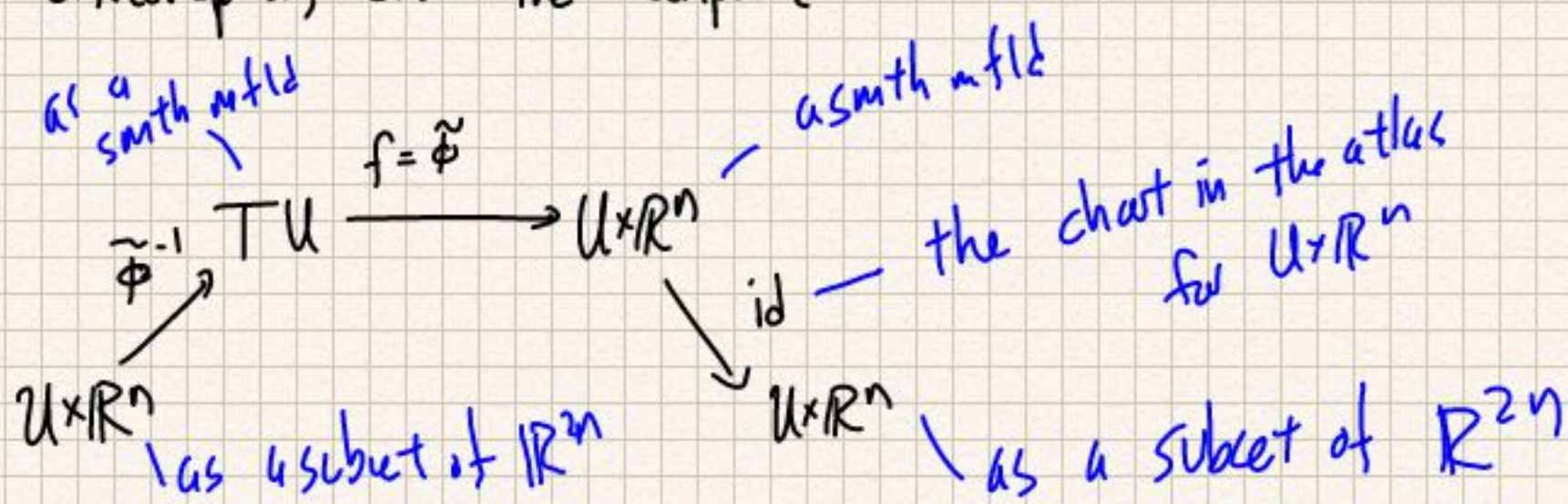
Pf TU has a single chart

$$\begin{array}{ccc} \tilde{\phi}: \bigcup_{p \in U} \{p\} \times T_p U & \cong & \bigcup_{p \in U} \{p\} \times T_p \mathbb{R}^n \\ \downarrow & & \\ & & U \times \mathbb{R}^n \end{array}$$

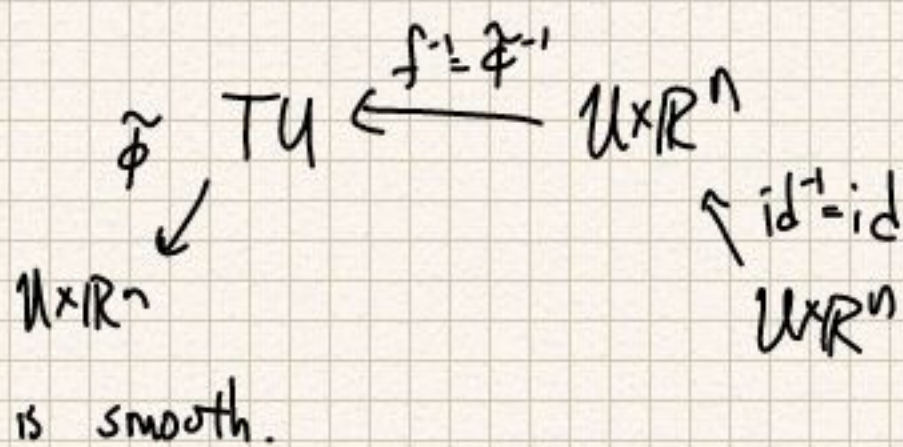
sending

$$(p, v) \mapsto (p, (v(x_i))_{i=1}^n).$$

This itself is a smooth diffeomorphism, since the composite



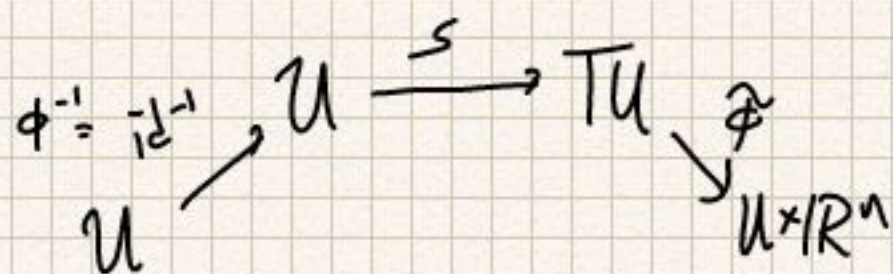
is obviously a smooth map, being $\text{id}_{U \times \mathbb{R}^n}$. Likewise,



That this is a bundle map is obvious since it commutes with projection, and is linear on fibers. //

$$\underline{\underline{\text{Cor}}} \quad \Gamma(TU) \cong C^\infty(U, \mathbb{R}^n).$$

PF A section $s: U \rightarrow TU$ is C^∞ iff



is C^∞ . Since s is a section,

we know the composite

$$U \longrightarrow U \times \mathbb{R}^n$$

is of the form

$$x \longmapsto (x, f_1(x), \dots, f_n(x))$$

ie, it determines a fn

$$f: U \longrightarrow \mathbb{R}^n$$

$$x \longmapsto (f_1(x), \dots, f_n(x)).$$

This is C^∞ iff the

composite

$$U \longrightarrow U \longrightarrow TU \longrightarrow U \times \mathbb{R}^n$$

is

//

Thm Let M be a smooth manifold. Then

$$T(TM) \cong \text{Der}(C^\infty(M), C^\infty(M))$$

Pf Let

$$\alpha: T(TM) \longrightarrow \text{Der}(C^\infty(M), C^\infty(M))$$

be as follows:

$$\text{Given } f \in C^\infty(M), s \in T(TM),$$

$$\text{let } \alpha(s)(f)(p) = s(p)(f)$$

the derivation we're defining
what the derivation does to f

evaluate $\alpha(s)(f)$ at $p \in M$.

A tangent vector to U at p_i
ie, a derivation to \mathbb{R} .
what the derivation $s(p)$ does to f . This is a real number.

By definition, the $a_i(p)$

are the #'s such that

$$\hat{\Phi}_\alpha(p, X_p) = \left(\Phi_\alpha(p), \sum_{i=1}^n a_i(\Phi_\alpha(p)) \frac{\partial}{\partial x_i} \Big|_{\Phi_\alpha(p)} \right)$$

so the assignment

$$p \longmapsto a_i(p)$$

is C^∞ . i.e., $a_i \in C^\infty(\Phi_\alpha(U))$. Thus

$$\Phi_\alpha(p) \xrightarrow{Xf \circ \Phi_\alpha^{-1}} \sum_{i=1}^n \left(a_i \frac{\partial (f \circ \Phi_\alpha^{-1})}{\partial x_i} \right) (\Phi_\alpha(p))$$

product of
 C^∞ fns on $\Phi_\alpha(U)$

is C^∞ . This simultaneously proves continuity

(since $Xf^{-1}(U) \cap U_\alpha$ is open) and smoothness

(since $\Phi_\alpha(U) \hookrightarrow M \rightarrow \mathbb{R}$ is C^∞ .)

So we've defined $\alpha: T(TM) \rightarrow \text{Der}_{\mathbb{R}}(C^\infty M, C^\infty M)$
 $X \longmapsto (f \longmapsto Xf)$.

Let

$$\Gamma(TU) \xleftarrow{\beta} \text{Der}(\mathcal{C}^\infty(U), \mathcal{C}^\infty(U))$$

be the following:

(a) Note $\forall p \in U$, the map

$$\begin{aligned} \text{ev}_p: \mathcal{C}^\infty(U) &\rightarrow \mathbb{R} \\ f &\mapsto f(p) \end{aligned}$$

is a ring map. Hence, given any

$$X \in \text{Der}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)),$$

the composite

$$\text{ev}_p \circ X: \mathcal{C}^\infty(M) \rightarrow \mathbb{R} \quad \text{is a derivation.}$$

$\downarrow \text{ev}_p$

(b) Hence $\forall p$, we have a map

$$\begin{aligned} \text{Der}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) &\rightarrow \text{Der}(\mathcal{C}^\infty(M), \mathbb{R}) \stackrel{\text{ev}_p}{=} T_p U. \\ X &\mapsto \text{ev}_p \circ X \end{aligned}$$

(c) Hence we have a map

$$\begin{aligned} \beta: \text{Der}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M)) &\rightarrow \text{Functions}(U, TU) \\ X &\mapsto \left(p \mapsto (p, \text{ev}_p \circ X) \right). \end{aligned}$$

pf

$$\text{ev}_p \circ X(fg)$$

$$= \text{ev}_p(Xf \cdot g + f \cdot Xg)$$

$$= Xf(p) \cdot g(p) + f(p) \cdot Xg(p)$$

$$= \text{ev}_p \circ X(f) \cdot g(p) + f(p) \cdot \text{ev}_p \circ X(g)$$

We need to show

$\beta(x)$ is a smooth
section.

(It's obviously a section since

$$U \xrightarrow{\beta(x)} TU \xrightarrow{\pi} U$$

sends p to p .)

This means we have to show

$$U \xrightarrow{id = \phi^{-1}} U \xrightarrow{\beta(x)} TU \xrightarrow{\tilde{\phi} = (id, (v(x_i))_{i=1}^n)} U \times \mathbb{R}^n$$

is smooth. This composite is:

$$p \xrightarrow{\quad} p \xrightarrow{\quad} (p, ev_p \circ X) \xrightarrow{\quad} (p, \underbrace{\left((ev_p \circ X)(x_i) \right)_{i=1}^n}_{\text{evaluate this derivative on the function } x_i : U \rightarrow \mathbb{R}})$$

derivation $C^\infty(U) \rightarrow \mathbb{R}$
the function X associates to the function x_i
evaluate this function @ p .

Since X maps C^∞ fns to C^∞ fns, $X(x_i)$ is smooth. Hence the assignment

$$U \xrightarrow{\quad} \mathbb{R} \\ p \xrightarrow{\quad} X(x_i)(p)$$

is smooth.

Thus the composite

$$U \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} \begin{array}{c} U \\ \rightarrow TU \\ \rightarrow U \times \mathbb{R}^n \end{array}$$

is a function

$$U \longrightarrow U \times \mathbb{R}^n$$

such that each component function

$$U \longrightarrow U \times \mathbb{R}^n \xrightarrow{\substack{j\text{th} \\ \text{component}}} \mathbb{R}$$

is smooth. This means the composite is indeed smooth.

$\implies \beta(x)$ is a smooth section of TU .

So indeed, we have a map

$$T^*(TM) \xleftarrow{\beta} \text{Der}(C^\infty(M), C^\infty(M))$$

$$\text{NTS: } \alpha \circ \beta = \text{id}$$

$$\beta \circ \alpha = \text{id}$$

Well, given $Y \in \text{Der}(C^\infty M, C^\infty(M))$,

$$\begin{aligned} (\alpha \circ \beta)(Y)(f) &= \alpha \left(p \mapsto \text{ev}_p \circ Y \right) (f) \\ &= \left(p \mapsto \beta(Y)_p(f) \right) \\ &= \left(p \mapsto (\text{ev}_p \circ Y)(f) \right) \\ &= \left(p \mapsto (Yf)_p \right) \\ &= Yf. \end{aligned}$$

$$\Rightarrow (\alpha \circ \beta)(Y) = Y.$$

Likewise, given $X: M \rightarrow TM$,

$$\begin{aligned} \underbrace{(\beta \circ \alpha)(X)_p}_{\substack{\in \\ T_p M}}(f) &= \text{ev}_p \circ \alpha(X)(f) \\ &= \alpha(X)(f)_p \\ &= X_p(f) \end{aligned}$$

$$\Rightarrow (\beta \circ \alpha)(X)_p = X_p \quad \forall p$$

$$\Rightarrow (\beta \circ \alpha)(X) = X. //$$