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Let's prove

Thm Let M be a smooth manifold. Then

$$\Gamma(TM) \cong \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$$

We first prove it for M an open subset of $\mathbb{R}^n$, where the proof will seem concrete.

How did we define TM?

Given an atlas, $A_M = \{(U_\alpha, \phi_\alpha)\}_\alpha$

we topologized TM by gluing together

$$\phi_\alpha(U_\alpha) \times \mathbb{R}^n$$

via transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} \times D(\phi_\beta \circ \phi_\alpha^{-1}).$$

Then

$$\tilde{U}_\alpha = \bigcup_{p \in U_\alpha} \{p\} \times T_p M$$

and

$$\tilde{\Phi}_\alpha : (x, v) \longmapsto (\phi_\alpha(x), v(x_i \circ \phi_\alpha))$$

defined an atlas $\{(\tilde{U}_\alpha, \tilde{\Phi}_\alpha)\}_\alpha$

for TM.

In a similar fashion,

let

$$(T_pM)^\vee := \hom_{\mathbb{R}}(T_pM, \mathbb{R}).$$

Thus one can give

$$T^*M := \bigcup_{p\in M} \{p\} \times T_pM^\vee$$

the structure of a smooth vector bundle. This is the <u>cotangent bundle</u> of M.

<u>Defn</u> A section of T^*M is called a smooth <u>1-form</u> on M.

<u>Rmk</u> $\exists$ map

$$\Gamma(T^*M) \times \Gamma(TM) \to C^\infty(M).$$

Exer (For home) Prove smooth structure of TM is independent of choice of atlas on M.

We saw TM is a vector bundle.

Defn Let $E \to M$ and $E' \to M'$ be smooth vector bundles.

A bundle map from E to E' is a smooth map $f: E \to E'$ s.t.

- $\exists$ a smooth map

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & E' \arrow[d] \\ M \arrow[r, dashed] & M' \end{tikzcd}$$

$\iff$ " u, v in same fiber $\Rightarrow$ $f(u), f(v)$ in same fiber "

making this diagram commute.

- $f|_{\pi^{-1}(p)}$ is a linear map $\forall p \in M$

A (smooth) bundle map f is called a (smooth) bundle isomorphism if f is a diffeomorphism.

in our class, smoothness is always assumed.

Exer (Home) If f is a diffeomorphism, then f^{-1} is a bundle map.

Prop Let $\mathcal{U} \subset \mathbb{R}^n$ be open, and given the standard atlas $\{(\mathcal{U}, \phi: \mathcal{U} \hookrightarrow \mathbb{R}^n)\}$. Then $\exists$ a bundle isomorphism

$$T\mathcal{U} \cong \mathcal{U} \times \mathbb{R}^n$$

Pf $T\mathcal{U}$ has a single chart

$$\begin{array}{c} \Phi: \bigcup_{p \in \mathcal{U}} \{p\} \times T_p\mathcal{U} \cong \bigcup_{p \in \mathcal{U}} \{p\} \times T_p\mathbb{R}^n \\ \downarrow \\ \mathcal{U} \times \mathbb{R}^n \end{array}$$

sending

$$(p, v) \longmapsto (p, (v(x_i))_{i=1}^n).$$

This Half is a smooth diffeomorphism, since the composite

as a smth mfld — a smth mfld

$$\begin{tikzcd} & TU \arrow[r, "f = \tilde{\phi}"] & U \times \mathbb{R}^n \arrow[d, "\mathrm{id}"] \\ U \times \mathbb{R}^n \arrow[ur, "\tilde{\phi}^{-1}"] & & U \times \mathbb{R}^n \end{tikzcd}$$

id — the chart in the atlas for $U \times \mathbb{R}^n$

$U \times \mathbb{R}^n$ as a subset of $\mathbb{R}^{2n}$ — $U \times \mathbb{R}^n$ as a subset of $\mathbb{R}^{2n}$

is obviously a smooth map, being $\mathrm{id}_{U \times \mathbb{R}^n}$. Likewise,

$$\begin{tikzcd} & TU \arrow[dl, "\tilde{\phi}"'] & U \times \mathbb{R}^n \arrow[l, "f^{-1} = \tilde{\phi}^{-1}"'] \\ U \times \mathbb{R}^n & & U \times \mathbb{R}^n \arrow[u, "\mathrm{id}^{-1} = \mathrm{id}"'] \end{tikzcd}$$

is smooth.

That this is a bundle map is obvious since it commutes with projection, and is linear on fibers. //

<u>Cor</u> $\Gamma(TU) \cong C^\infty(U, \mathbb{R}^n)$.

<u>Pf</u> A section $s : U \to TU$ is C^∞ iff

$$\begin{tikzcd} & U \arrow[r, "s"] & TU \arrow[dr, "\tilde{\phi}"] & \\ U \arrow[ur, "\phi^{-1} = \mathrm{id}^{-1}"] & & & U \times \mathbb{R}^n \end{tikzcd}$$

is C^∞. Since s is a section,

we know the composite

$$\mathcal{U} \longrightarrow \mathcal{U} \times \mathbb{R}^n$$

is of the form

$$x \longmapsto (x, f_1(x), \cdots f_n(x))$$

ie, it determines a fxn

$$f : \mathcal{U} \longrightarrow \mathbb{R}^n$$
$$x \longmapsto (f_1(x), \cdots, f_n(x)).$$

This is C^∞ iff the composite

$$\mathcal{U} \longrightarrow \mathcal{U} \longrightarrow T\mathcal{U} \longrightarrow \mathcal{U} \times \mathbb{R}^n$$

is. //

<u>Thm</u> Let M be a smooth manifold. Then

$$\Gamma(TM) \cong \mathrm{Der}(C^\infty(M), C^\infty(M))$$

<u>Pf</u> Let

$$\alpha : \Gamma(TM) \longrightarrow \mathrm{Der}(C^\infty(M), C^\infty(M))$$

be as follows:

Given $f \in C^\infty(M)$, $s \in \Gamma(TM)$,

let $\alpha(s)(f)(p) = s(p)(f)$

A tangent vector to $\mathcal{U}$ at p; ie, a derivation to $\mathbb{R}$.

the derivation we're defining

what the derivation does to f

evaluate $\alpha(s)(f)$ at $p \in \mathcal{U}$.

what the derivation $s(p)$ does to f. This is a real number.

i.e.,

$$\alpha : X \longmapsto (f \mapsto Xf)$$

Why is $p \overset{Xf}{\longmapsto} X_p f$ C^∞?

NTS: $\forall\ U \subset \mathbb{R}$, $Xf^{-1}(U)$ is open. Well,

$$Xf^{-1}(U) = \bigcup_\alpha Xf\big|_{U_\alpha}^{-1}(U)$$

smth in local coords

So we're done if $\left(Xf\big|_{U_\alpha}\right)^{-1}(U)$ is open $\forall\ \alpha$.

Well,

$$\begin{tikzcd} & U_\alpha \arrow[r, hook] & M \arrow[r, "Xf"] & \mathbb{R} \supset U \arrow[d, "id"] \\ \phi(U_\alpha) \arrow[ur, "\phi_\alpha^{-1}"] & & & U \end{tikzcd}$$

is the map

$$\begin{tikzcd} & p \arrow[r, mapsto] & Xf(p) = X_p f \\ \phi(p) \arrow[ur, mapsto] \end{tikzcd}$$

$\frac{\partial}{\partial x_i}\big|_p \in T_p(\phi_\alpha(U_\alpha))$

but $X_p(f) = \left(\sum \underbrace{a_i(p)}_{\text{some \#'s}} \tilde{\Phi}_\alpha^{-1}\left(\frac{\partial}{\partial x_i}\Big|_p\right)\right)(f)$

$= \sum a_i(p) \frac{\partial}{\partial x_i}\Big|_p \left(f \circ \phi_\alpha^{-1}\right)$ by defn of $\tilde{\Phi}_\alpha$, and proof that $\frac{\partial}{\partial x_i}\big|_p$ span $T_p\mathbb{R}^n$?

$= \sum_{i=1}^{n} a_i(p) \frac{\partial (f \circ \phi_\alpha^{-1})}{\partial x_i}.$

By definition, the $a_i(p)$

are the #'s such that

$$\tilde{\Phi}_\alpha(p, X_p) = \left(\phi_\alpha(p), \sum_{i=1}^{n} a_i(\phi_\alpha(p)) \frac{\partial}{\partial x_i}\bigg|_{\phi_\alpha(p)}\right)$$

so the assignment

$$p \longmapsto a_i(p)$$

is C^∞. ie, $a_i \in C^\infty(\phi_\alpha(U))$. Thus

$$\phi_\alpha(p) \xmapsto{Xf \circ \phi_\alpha^{-1}} \sum_{i=1}^{n} \Bigg(\underbrace{a_i \frac{\partial (f \circ \phi_\alpha^{-1})}{\partial x_i}}_{\substack{\text{product of} \\ C^\infty \text{ fxns on } \phi_\alpha(U)}}\Bigg)(\phi_\alpha(p))$$

is C^∞. This simultaneously proves continuity (since $Xf^{-1}(U) \cap U_\alpha$ is open) and smoothness (since $\phi_\alpha(U) \hookrightarrow M \to \mathbb{R}$ is C^∞.)

So we've defined

$$\alpha : \Gamma(TM) \longrightarrow \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$$
$$X \longmapsto (f \longmapsto Xf).$$

Let

$$\Gamma(TU) \xleftarrow{\beta} \mathrm{Der}(C^\infty(U), C^\infty(U))$$

be the following:

(a) Note $\forall p \in U$, the map

$$\begin{aligned} ev_p : C^\infty(U) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(p) \end{aligned}$$

is a ring map. Hence, given any

$$X \in \mathrm{Der}(C^\infty(M), C^\infty(M)),$$

the composite

$$ev_p \circ X : C^\infty(M) \longrightarrow \mathbb{R} \quad \text{is a derivation.}$$

$\circlearrowleft_{ev_p}$

Pf

$ev_p \circ X(fg)$

$= ev_p(Xf \cdot g + f \cdot Xg)$

$= Xf(p) \cdot g(p) + f(p) \cdot Xg(p)$

$= ev_p \circ X(f) \cdot g(p) + f(p) \cdot ev_p \circ Xg$

(b) Hence $\forall p$, we have a map

$$\begin{aligned} \mathrm{Der}(C^\infty(M), C^\infty(M)) &\longrightarrow \mathrm{Der}(C^\infty(M), \mathbb{R}^{\circlearrowleft ev_p}) =: T_p U. \\ X &\longmapsto ev_p \circ X \end{aligned}$$

(c) Hence we have a map

$$\begin{aligned} \beta : \mathrm{Der}(C^\infty(M), C^\infty(M)) &\longrightarrow \mathrm{Functions}(U, TU) \\ X &\longmapsto \big(p \mapsto (p, ev_p \circ X) \big). \end{aligned}$$

We need to show

$\beta(X)$ is a <u>smooth</u> section.

(It's obviously a section since

$$\mathcal{U} \xrightarrow{\beta(X)} T\mathcal{U} \xrightarrow{\pi} \mathcal{U}$$

sends p to p.)

This means we have to show

$$\begin{tikzcd}
 & \mathcal{U} \arrow[r, "\beta(X)"] & T\mathcal{U} \arrow[d, "\tilde{\phi} = (\mathrm{id},\ (v(x_i))_{i=1}^n)"] \\
\mathcal{U} \arrow[ur, "\mathrm{id} = \phi^{-1}"] & & \mathcal{U} \times \mathbb{R}^n
\end{tikzcd}$$

is smooth. This composite is:

$$\begin{tikzcd}
 & p \arrow[r, mapsto] & (p,\ \mathrm{ev}_p \circ X) \arrow[d, mapsto] \\
p \arrow[ur, mapsto] & & \left(p,\ \Big(\underbrace{\overbrace{(\mathrm{ev}_p \circ X)}^{\text{derivation } C^\infty(\mathcal{U}) \to \mathbb{R}} (x_i)}_{\text{evaluate this derivation on the function } x_i : \mathcal{U} \to \mathbb{R}.} \Big)_{i=1}^n \right)
\end{tikzcd}$$

Fixing i, we see

$$(\mathrm{ev}_p \circ X)(x_i) = \underbrace{\overbrace{X(x_i)}^{\text{the function } X \text{ associates to the function } x_i} (p)}_{\text{evaluate this function @ } p.}$$

Since X maps C^∞ fxns to C^∞ fxns, $X(x_i)$ is smooth. Hence the assignment

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathbb{R} \\ p &\longmapsto X(x_i)(p) \end{aligned}$$

is smooth.

Thus the composite

$$\begin{tikzcd} & U \arrow[r] & TU \arrow[dr] & \\ \mathcal{U} \arrow[ur] & & & U \times \mathbb{R}^n \end{tikzcd}$$

is a function

$$U \longrightarrow U \times \mathbb{R}^n$$

such that each component function

$$\mathcal{U} \longrightarrow \mathcal{U} \times \mathbb{R}^n \xrightarrow[\substack{i^{th} \\ \text{component}}]{} \mathbb{R}$$

is smooth. This means the composite is indeed smooth.

$\Longrightarrow$ $\beta(x)$ is a smooth section of TU.

So indeed, we have a map

$$\Gamma(TM) \xleftarrow{\ \beta\ } \mathrm{Der}(C^\infty(M), C^\infty(M))$$

NTS: $\alpha \circ \beta = \mathrm{id}$

$\beta \circ \alpha = \mathrm{id}$.

Well, given $Y \in \mathrm{Der}(C^\infty(M), C^\infty(M))$,

$$(\alpha\circ\beta)(Y)(f) = \alpha\big(\overbrace{p \mapsto ev_p \circ Y}^{\beta(Y)}\big)(f)$$

$$= \big(p \longmapsto \beta(Y)_p(f)\big)$$

$$= \big(p \longmapsto (ev_p \circ Y)(f)\big)$$

$$= \big(p \longmapsto (Yf)(p)\big)$$

$$= Yf.$$

$$\Rightarrow \quad (\alpha\circ\beta)(Y) = Y.$$

Likewise, given $X : M \longrightarrow TM$,

$$\underbrace{(\beta\circ\alpha)(X)(p)}_{\in\, T_pM}(f) = ev_p \circ \alpha(X)(f)$$

$$= \alpha(X)(f)(p)$$

$$= X_p(f)$$

$$\Rightarrow \quad (\beta\circ\alpha)(X)(p) = X(p) \quad \forall p$$

$$\Rightarrow \quad (\beta\circ\alpha)(X) = X. \quad //$$