Mon. Sept 15, 2014

Today we'll be talking about Lie algebras. Like derivations, they are an algebraic entity with great geometric examples. With time, you might start thinking of every Lie algebra as coming from some group (broadly construed).

Defn Let k be a field.
A Lie algebra over k is a vector space L over k, together with a bilinear map

$$L \times L \longrightarrow L$$
$$(x, y) \longmapsto [x, y]$$

$[ax_1 + x_2, y] = a[x_1, y] + [x_2, y]$ $\forall a \in k,\ x, y \in L$

such that

(i) $\forall x, y \in L$, $[x,y] = -[y,x]$ antisymmetric

(ii) $\forall x, y, z \in L$,
$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$ Jacobi identity

Rmk Let $[x,-] : L \longrightarrow L$; write it D_x. Then the Jacobi identity says

$$D_x[y,z] = [D_x y, z] + [y, D_x z]$$

Just like

$$D(fg) = Df \cdot g + f \cdot Dg.$$

ie, the Jacobi identity says $[x,-]$ is a derivation with respect to the operation $[-,-]$.

Ex Let $M_{n\times n}$ be the set of $n\times n$ matrices over k. Then we have a bilinear map

$$M_{n\times n}\times M_{n\times n}\longrightarrow M_{n\times n}$$
$$(A,B)\longmapsto AB-BA.$$

(i) antisymmetry is obvious.

(ii) $[A,[B,C]] = A(BC-CB)-(BC-CB)A$ ①

$[[A,B],C] = (AB-BA)C - C(AB-BA)$ ②

$[B,[A,C]] = B(AC-CA) - (AC-CA)B$ ③

Then ②+③=①. So Jacobi identity is satisfied.

Ex More generally, if R is any k-algebra, the operation
$$R\times R\longrightarrow R$$
$$(A,B)\longmapsto AB-BA$$
turns R into a Lie algebra over k. (Same proof as above.)

Moreover:

Prop Let R be a k-algebra. Then

$$\mathrm{Der}_k(R,R)\times \mathrm{Der}_k(R,R)\longrightarrow \mathrm{Der}_k(R,R)$$
$$(X,Y)\longmapsto X\circ Y-Y\circ X$$

is a Lie algebra. (i.e., $\mathrm{Der}_k(R,R)$ is a sub-Lie algebra of $\hom_k(R,R)$.)

Cor $\Gamma(TM)$ is
a Lie algebra over $\mathbb{R}$.

We'll explore the bracket of two vector fields in a moment.

First, a proof.

Pf (of Prop'n) We just need to verify that $X \circ Y - Y \circ X \in \mathrm{Der}_k(R,R)$.

Let $f, g \in R$. Then

$$X(Y(fg)) - Y(X(fg))$$

$$= X(Yf \cdot g + f \cdot Yg) - Y(Xf \cdot g + f \cdot Xg)$$

$$= XYf \cdot g + \cancel{Xf \cdot Yg} - YXf \cdot g - \cancel{Yf \cdot Xg}$$
$$+ \cancel{Yf \cdot Xg} + f \cdot XYg - \cancel{Xf \cdot Yg} - f \cdot YXg$$

$$= (XY - YX)f \cdot g + f \cdot (XY - YX)g. \quad //$$

What the dickens is this?

Recall:

Given any $X \in \Gamma(TM)$,
we have a map

$$X_p : C^\infty(M) \longrightarrow \mathbb{R}$$
$$f \longmapsto X_p(f)$$

ie, take directional derivative of f at p, in direction of X_p.

Doing this $\forall$ p, we have

$$X : C^\infty(M) \longrightarrow C^\infty(M)$$
$$f \longmapsto (p \mapsto X_p(f))$$

Check the assignment $p \mapsto X_p(f)$ is smooth in $p \in M$ using local coords

Thm The map

$$\Gamma(TM) \longrightarrow \mathrm{Der}(C^\infty(M), C^\infty(M))$$

is an isomorphism of $C^\infty(M)$-modules.

So what does $[X,Y] \in \Gamma(TM)$ look like in local coordinates? Let $X = \sum X_i \frac{\partial}{\partial x_i}$, $Y = \sum Y_i \frac{\partial}{\partial x_i}$.

X_i a C^∞ fxn on $\mathbb{R}^n$

C^∞ fxn on $\mathbb{R}^n$

A function $f \in C^\infty(M)$ is sent to

$$\begin{aligned}
[X,Y]f &= X(Yf) - Y(Xf) \\
&= X\left(\sum Y_i \frac{\partial f}{\partial x_i}\right) - Y\left(\sum X_j \frac{\partial f}{\partial x_j}\right) \\
&= \sum\sum X_j \frac{\partial}{\partial x_j}\left(Y_i \frac{\partial f}{\partial x_i}\right) - Y_i \frac{\partial}{\partial x_i}\left(X_j \frac{\partial f}{\partial x_j}\right) \\
&= \sum\sum X_j \frac{\partial Y_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \underbrace{X_j Y_i \frac{\partial^2 f}{\partial x_i \partial x_j} - Y_i X_j \frac{\partial^2 f}{\partial x_i \partial x_j}}_{=0} - Y_i \frac{\partial X_j}{\partial x_i}\frac{\partial f}{\partial x_j}.
\end{aligned}$$

So in local coords,

$$[X,Y] = \sum_{i,j} X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial x_i}$$

$$= \sum_{i,j=1}^{n} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} .$$

Why would you care about

- finite-dimensional Lie algebras?

  ⇝ Study of Lie groups reduces to study of of Lie algebras

- Lie algebra of vector fields?

  ⇝ algebraic characterization of submanifolds of M (Frobenius theorem)

  ⇝ Studying the ∞-dim Lie group, $\mathrm{Diff}(M)$.

  ↑ group of diffeomorphisms of M.

Defn Let
$$f: M \longrightarrow N$$
be smooth. We say two vector fields
$$X \in \Gamma(TM),\ Y \in \Gamma(TN)$$
are *f-related* if the diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ X \uparrow & & \uparrow Y \\ M & \xrightarrow[f]{} & N \end{array}$$

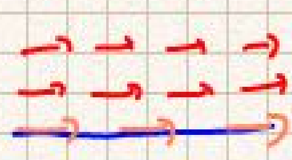
commutes.

Intuition: Vector fields rarely "push forward," nor naturally "pull back." (You can take away quotation marks if you're familiar w/ the terms/notions). At the least it requires choices.

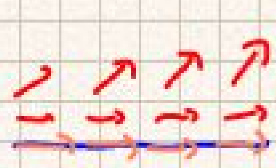
Ex If $M \longrightarrow N$ isn't injective.

No pushforward!

Ex If $M \hookrightarrow N$ injective, many choices of defining a vec. field on all of



or



Likewise for pulling back, if $M \to N$ not injective. (Ex: If $M \to N$ is a submersion.)

If X happens to look like a pullback of Y, or if Y happens to be (an extension of) a pushforward of a vector field on M, then the two are f-related.

Prop'n Fix $f: M \to N$.
If Y_i is f-related to X_i, $i=1,2$, then

$$[Y_1, Y_2]$$

is f-related to

$$[X_1, X_2].$$

Pf. By defn, $(Y_1)_{f(p)} = T_pf((X_1)_p)$

so $\forall\ h: N \to \mathbb{R}$ smooth,

$$(Y_1)_{f(p)}(h) = (X_1)_p(h \circ f).$$

ie, $(Y_1 h)(f(p)) = X_1(h\circ f)(p)\ \forall p \in M.$

Then

$$\begin{aligned}[Y_2, Y_1](h)(f(p)) &= (Y_2(Y_1(h)) - Y_1(Y_2 h))(f(p)) \\ &= (X_2((Y_1 h)\circ f) - X_1((Y_2 h)\circ f))(p) \\ &= (X_2(X_1(h\circ f)) - X_1(X_2(h\circ f)))(p) \\ &= [X_2, X_1](h\circ f)(p).\end{aligned}$$

Since this is true $\forall\ h$, $[Y_2, Y_1]_{f(p)} = T_p([X_2, X_1])\ \forall p$.

ie,
$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[dr, phantom, "\circlearrowright"] & TN \\ M \arrow[u, "{[X_2,X_1]}"] \arrow[r] & N \arrow[u, "{[Y_2,Y_1]}"'] \end{tikzcd}$$

//

Intuitively, let's let
$$M \xhookrightarrow{f} N$$
be a smooth, injective map for which T_pf is injective $\forall p \in M$.

Then looking at all pairs of f-related vector fields (X, Y), the collection of Y should determine a set of vector fields which are all closed under the Lie bracket (when restricted to $f(M)$).

How about the converse?

When can a certain collection closed under Lie bracket determine a submanifold?

This is the subject of the Frobenius theorem.

But next time, we'll talk about cotangent bundles and differential forms.