Friday, Sept 12, 2014

Homework two now due on 22^{nd} of September. (We're moving slower.)

Recall: $T_0\mathbb{R}^n \cong \mathbb{R}^n$,

spanned by

$$\left\{ \frac{\partial}{\partial x_i}\bigg|_{x=0} \right\} \in \mathrm{Der}\left(C^\infty(\mathbb{R}^n), \mathbb{R}\right)_{ev_0}$$

<u>Cor</u> $\left\{ \frac{\partial}{\partial x_i}\bigg|_{x} \right\}$ spans $T_x\mathbb{R}^n$.

$$C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

$$f \longmapsto \frac{\partial f}{\partial x_i}(x).$$

We defined, $\forall\, f: M \to N$ smooth,

$$T_p f : T_p M \longrightarrow T_{f(p)} N$$
$$v \longmapsto v(- \circ f).$$

So $T_p f(v) \in \mathrm{Der}(C^\infty(N), \mathbb{R})$ is the composite

$$C^\infty(N) \xrightarrow{f^*} C^\infty(M) \xrightarrow{v} \mathbb{R}.$$

<u>Exer</u> Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be smooth. Show that

$$T_x f : T_x \mathbb{R}^m \to T_{f(x)} \mathbb{R}^n$$

sends
$$\underbrace{\sum_{i=1}^m a_i \frac{\partial}{\partial x_i}\bigg|_x}_{\in\, T_x\mathbb{R}^m} \longmapsto \sum_{k=1}^n (Df)_{kj}\, a_j \frac{\partial}{\partial x_k}\bigg|_{f(x)}$$

matrix of partial derivatives from multivariable calculus.

Pf Consider the fuction

$$y_k : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(y_1, \cdots, y_n) \longmapsto y_k .$$

Recall the isomorphism

$$T_0\mathbb{R}^n \cong \mathbb{R}^n$$

is given by

$$v \longmapsto \left(v(x_i)\right)_{i=1}^n .$$

ie,

$$v = \sum v(x_i) \frac{\partial}{\partial x_i}\Big|_{x=0} . \qquad (*)$$

So suffies to evaluate

$$(T_x f)\left(\sum a_i \frac{\partial}{\partial x_i}\right)(y_k) = \sum a_i \frac{\partial}{\partial x_i}(y_k \circ f)$$

$$= \sum a_i \, (Df)_{ki} .$$

By $(*)$, we have

$$(T_x f)\left(\sum a_i \frac{\partial}{\partial x_i}\right) = \sum a_i (Df)_{ki} \frac{\partial}{\partial y_k} .$$

//

Prop $\widetilde{\Phi}$ define

C^∞ atlas for TM.

Pf

$$(p,v) \longmapsto (\phi(p), (v(x_i\circ\phi))_i)$$

$$\begin{tikzcd}
\bigcup_{p\in U} \{p\}\times T_pM \arrow[r, "\widetilde{\Phi}"] \arrow[d] & \phi(U)\times\mathbb{R}^n \\
\bigcup_{x\in\phi(U)} \{x\}\times T_x\mathbb{R}^n \arrow[ur] &
\end{tikzcd}$$

$$(p,v) \longmapsto (\phi(p), T_p\phi(v)) \longmapsto (\phi(p), T_p\phi(v)(x_i))$$

So $\widetilde{\Phi}_\beta \circ \widetilde{\Phi}_\alpha^{-1}$ is given by

$$\begin{tikzcd}
\phi(U)\times\mathbb{R}^n \arrow[r] & \bigcup \{x\}\times T_x\mathbb{R}^n \arrow[r] \arrow[dr, "T_x(\phi_\beta\circ\phi_\alpha^{-1})"'] & \bigcup\{p\}\times T_pM \arrow[d] \\
 & & \bigcup\{y\}\times T_y\mathbb{R}^n \arrow[d] \\
 & & \phi_\beta(U)\times\mathbb{R}^n
\end{tikzcd}$$

ie,
$$\left(x, \sum a_i \frac{\partial}{\partial x_i}\Big|_x\right) \longmapsto \left(\phi_\beta\circ\phi_\alpha^{-1}(x),\ D(\phi_\beta\circ\phi_\alpha^{-1})(\vec{a})\right). \quad //$$

C^∞ by defn of atlas

C^∞ since $\phi_\beta\circ\phi_\alpha^{-1}$ is.

To topologize:

$$TM := \coprod_\alpha \phi_\alpha(U_\alpha) \times \mathbb{R}^n \Big/ \sim_{\text{charts}}$$

Can check $\tilde{\mathcal{U}}_\alpha \subset TM$ are open.

<u>Prop</u> The map

$$\begin{aligned} TM &\xrightarrow{\pi} M \\ (x, v) &\longmapsto x \end{aligned}$$

is smooth.

<u>Pf</u> First check it's continuous.

Well,

$$\pi^{-1}(U) = \tilde{\mathcal{U}} \leftarrow \text{open}$$

$\forall$ small open balls $U \subset M$.

To check smoothness:

$$\psi \circ \pi \circ \tilde{\phi}^{-1}$$

sends

$$(x, a) \mapsto (\phi^{-1}(x), T_x\phi^{-1}(a))$$

$$\xmapsto{\pi} \phi^{-1}(x)$$

$$\xmapsto{\psi} \psi \circ \phi^{-1}(x)$$

is C^∞ by defn of smooth atlas

on M. //

Defn Let M be a smth mfld. A real vector bundle of rank n is the data of:

- A mfld E and a smooth map $E \xrightarrow{\pi} M$
- $\forall x \in M$, the structre of a n-dim real vector space on $\pi^{-1}(x)$

such that for all $x \in X$, $\exists$ $U \subset X$ open, $x \in U$, and a diffeomorphism

$$\pi^{-1}(U) \xrightarrow{\phi} U \times \mathbb{R}^n$$

such that

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}_1"] \\
& U &
\end{tikzcd}$$

and $\phi|_{\pi^{-1}(\{y\})} : \pi^{-1}(\{y\}) \to \{y\} \times \mathbb{R}^n$

is a linear isomorphism.

Defn

- A <u>section</u> of a vector bundle is a map
$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$
such that $\pi \circ s = id_M$.

- A smooth map of smooth vector bundles $E_1 \to M_1$, $E_2 \to M_2$ is a commutative diagram
$$\begin{tikzcd} E_1 \arrow[r, "\bar{f}"] \arrow[d] & E_2 \arrow[d] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$
s.t. $\bar{f}, f$ are smooth, and $\bar{f}\big|_{\pi^{-1}\{x\}} : \pi^{-1}(\{x\}) \to \pi^{-1}(\{f(x)\})$ is a linear map.

Defn The set of smooth sections of $E \to M$ is denoted $\Gamma(E)$.

Note it is a module over $C^\infty(M)$, and hence over $\mathbb{R}$.

Ex TM is a vector bundle. Any smooth map

$$f: M_1 \longrightarrow M_2$$

induces a bundle map

$$Tf: TM_1 \longrightarrow TM_2$$
$$(x, v) \longmapsto (f(x), T_x f(v)).$$